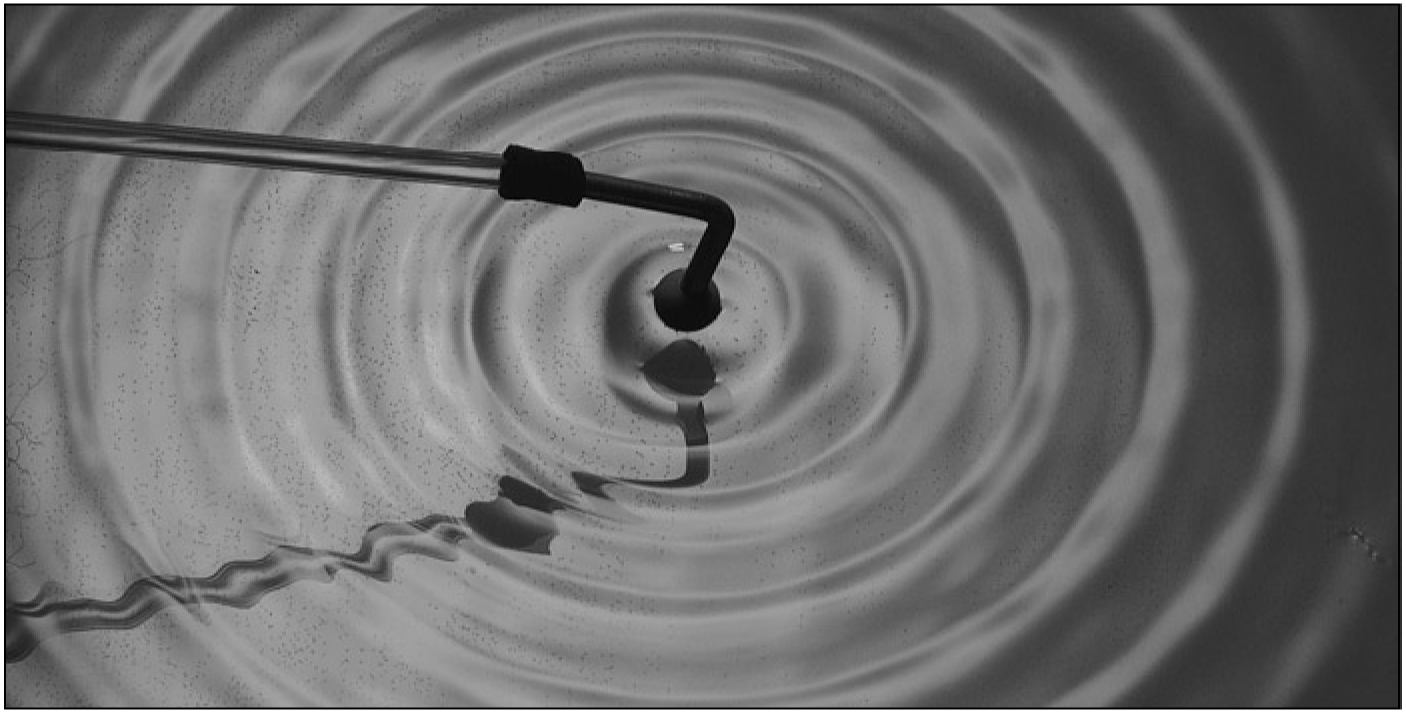
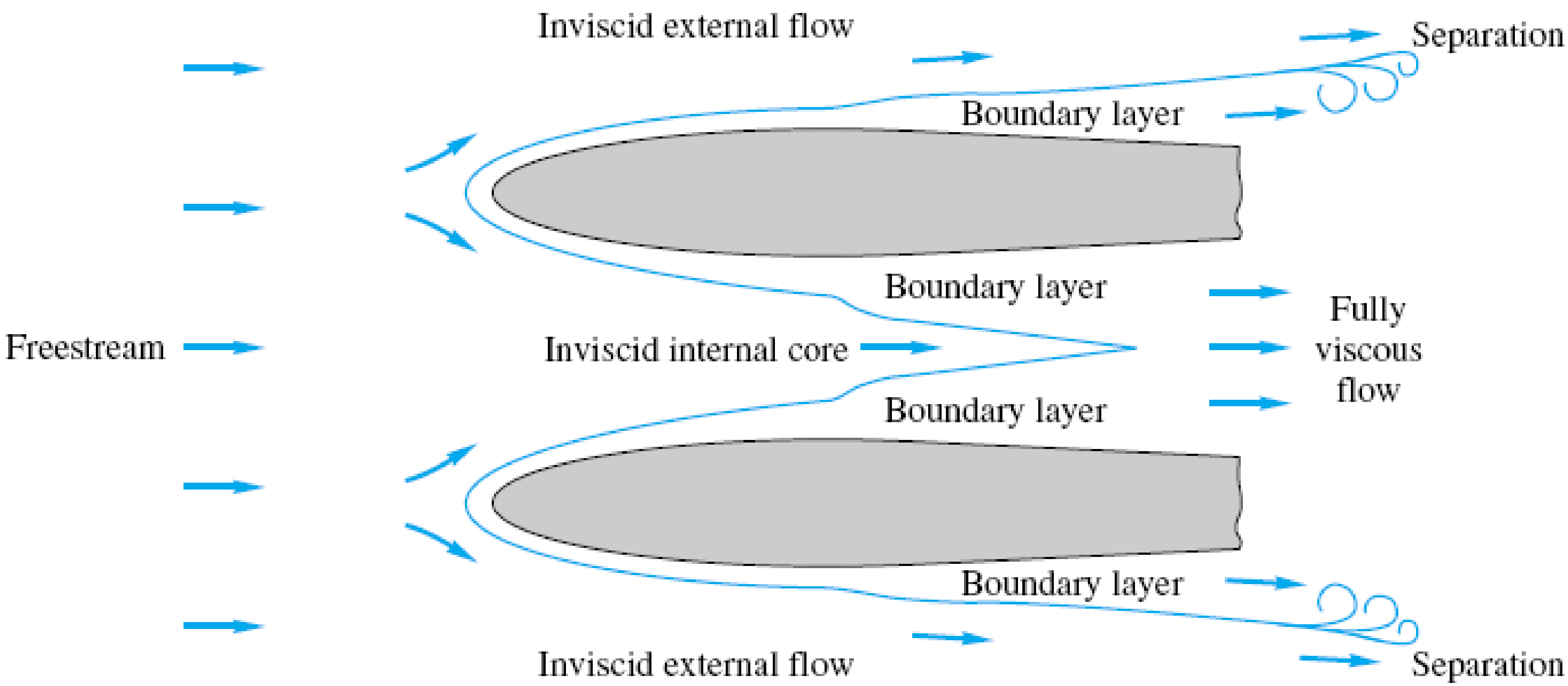
فصل (چهارم) هشتم: جریان پتانسیل



Cylindrical wave pattern produced in a ripple tank. When not modified by the no-slip condition at solid surfaces, waves are nearly inviscid and well represented by the potential theory of this chapter. *(Courtesy of Dr. E. R. Degginger/Color-Pic Inc.)*

فصل هشتم: جریان پتانسیل

8.1 Introduction and Review



فصل هشتم: جریان پتانسیل

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

If $\nabla \times \mathbf{V} \equiv 0$ then $\mathbf{V} = \nabla \phi$

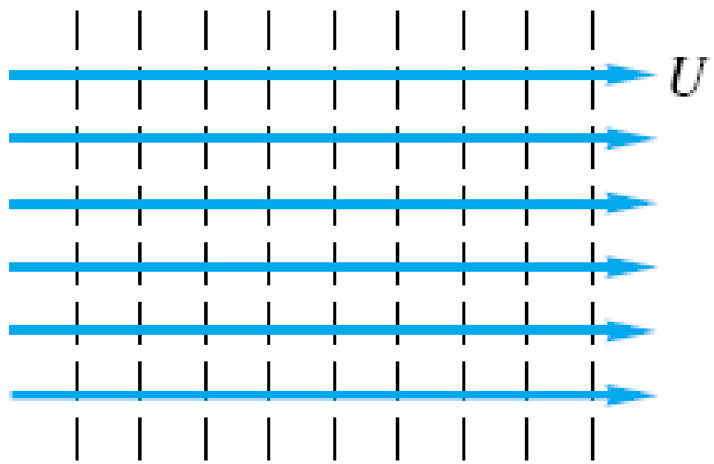
$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

4.10 Some Illustrative Plane Potential Flows

Uniform Stream in the x Direction

$$\mathbf{V} = \mathbf{i}U, \qquad u = U = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Uniform stream $\mathbf{i}U$: $\quad \psi = Uy \qquad \phi = Ux$

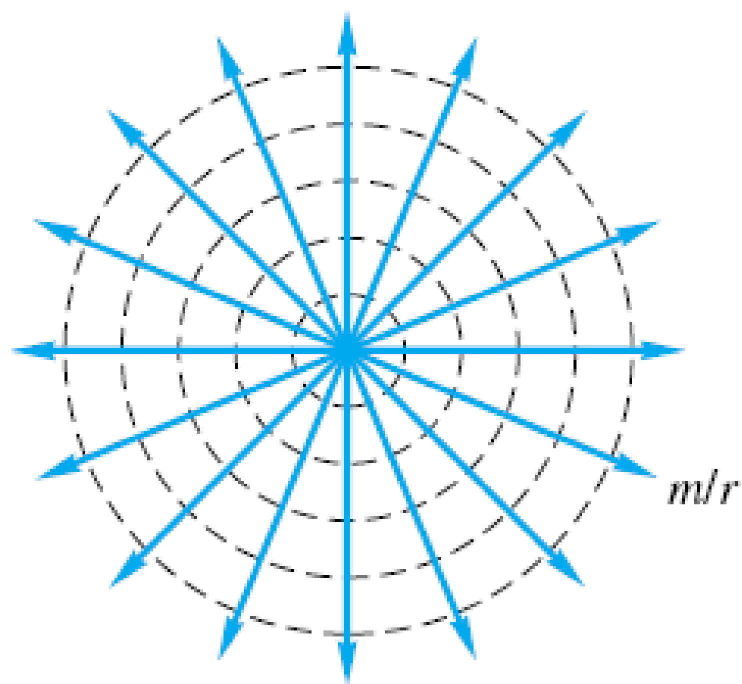


Solid lines are streamlines;
dashed lines are potential lines.

Line Source or Sink at the Origin

$$v_r = \frac{Q}{2\pi r b} = \frac{m}{r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} \qquad v_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Line source or sink: $\psi = m\theta \qquad \phi = m \ln r$



Solid lines are streamlines;
dashed lines are potential lines.

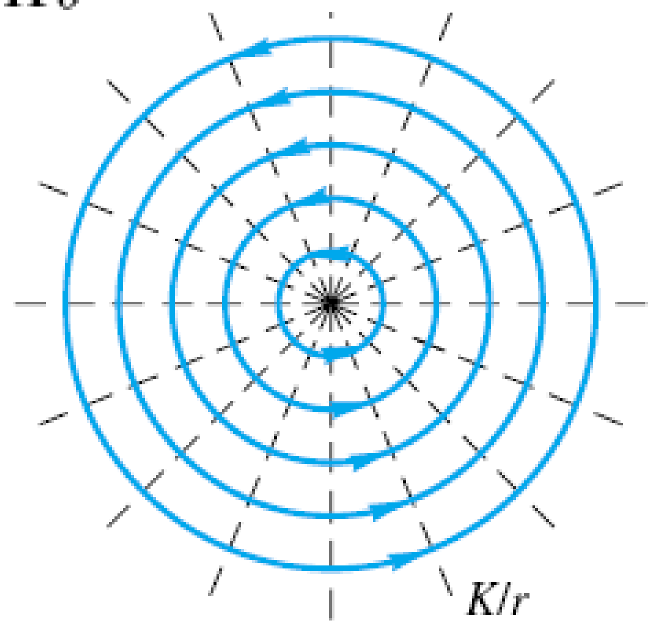
Line Irrotational Vortex

that only one function $v_\theta(r)$ is *irrotational*, i.e., curl $\mathbf{V} = 0$, and that is $v_\theta = K/r$, This is sometimes called a *free vortex*,

$$v_\theta = \frac{K}{r} = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad v_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

$$\psi = -K \ln r \qquad \phi = K\theta$$

K is a constant called the *strength* of the vortex.



Solid lines are streamlines;
dashed lines are potential lines.

Superposition: Source Plus an Equal Sink

an incompressible irrotational flow and therefore satisfies both plane "potential flow" equations $\nabla^2\psi = 0$ and $\nabla^2\phi = 0$.

Since these are linear partial differential equations, any *sum* of such basic solutions is also a solution.

consider a source $+m$ at $(x, y) = (-a, 0)$,

combined with a sink of equal strength $-m$, placed at $(+a, 0)$,

$$\psi = \psi_{\text{source}} + \psi_{\text{sink}} = m \tan^{-1} \frac{y}{x+a} - m \tan^{-1} \frac{y}{x-a}$$

$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} = \frac{1}{2} m \ln [(x+a)^2 + y^2] - \frac{1}{2} m \ln [(x-a)^2 + y^2]$$

Source plus sink:

$$\psi = -m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2} \qquad \phi = \frac{1}{2} m \ln \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \tag{4.133}$$

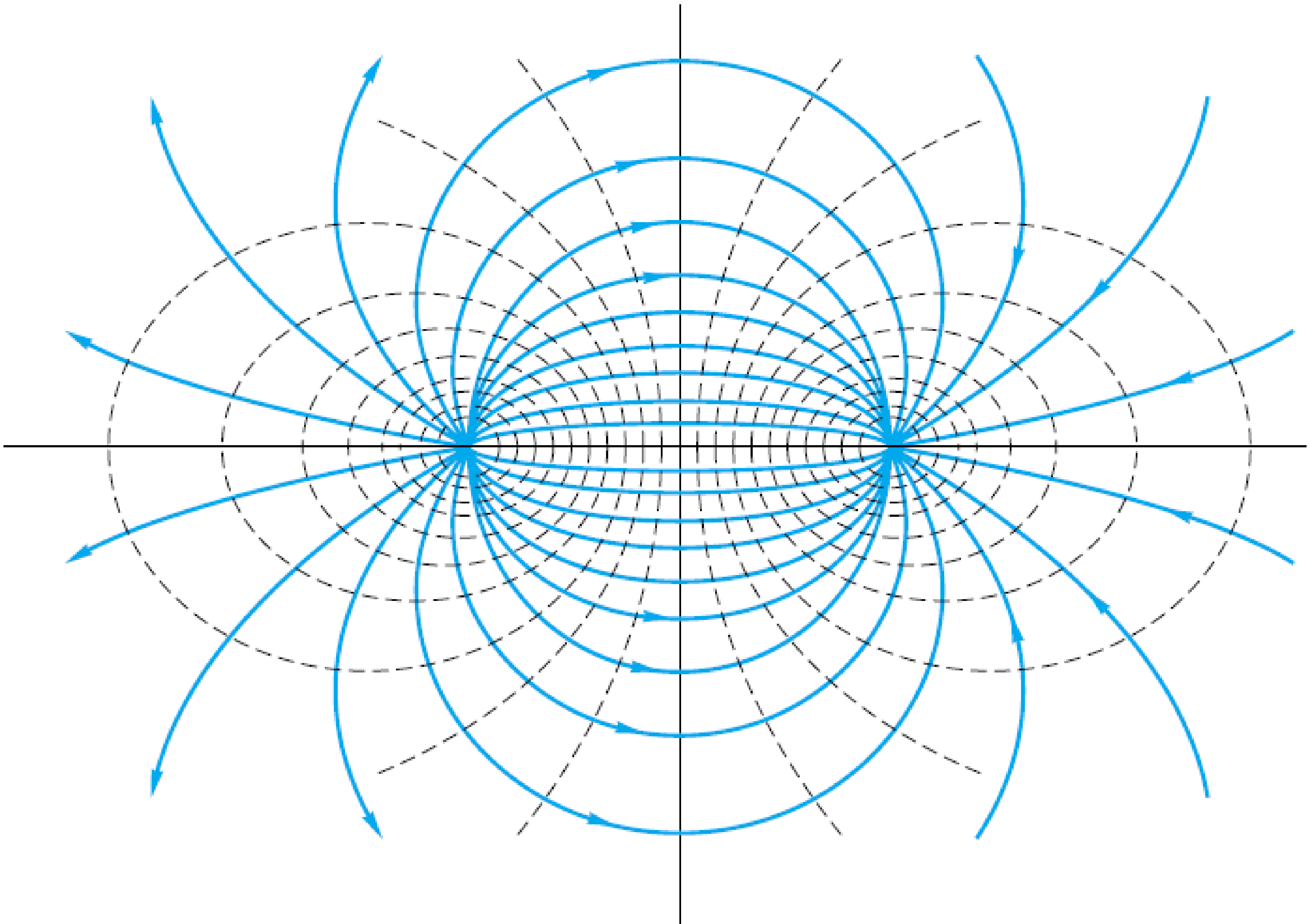


Fig. 4.13 Potential flow due to a line source plus an equal line sink, from Eq. (4.133). Solid lines are streamlines; dashed lines are potential lines.

Sink Plus a Vortex at the Origin

Sink plus vortex: $\psi = m\theta - K \ln r \qquad \phi = m \ln r + K\theta \qquad (4.134)$

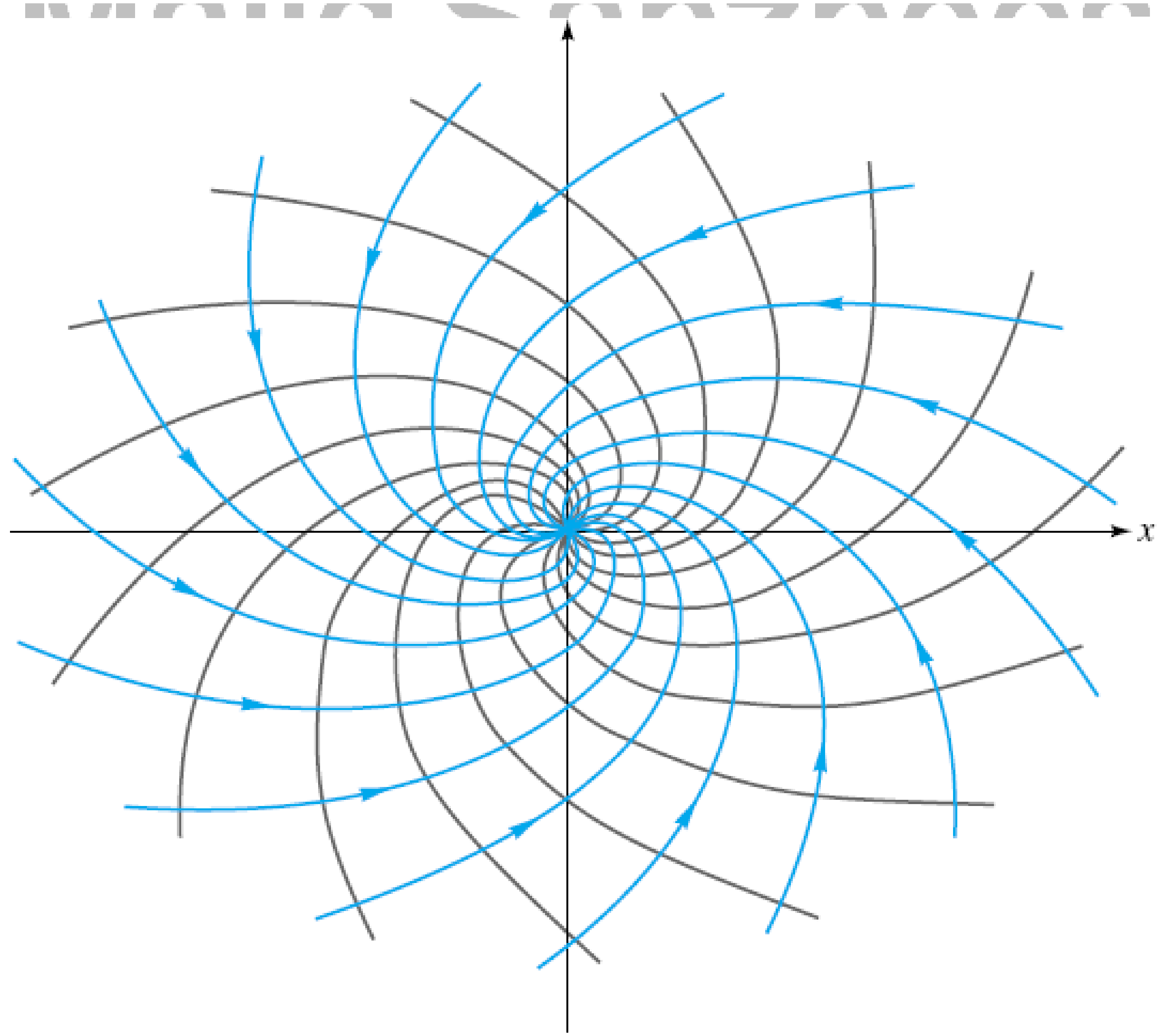


Fig. 4.14 Superposition of a sink plus a vortex, Eq. (4.134), simulates a tornado.

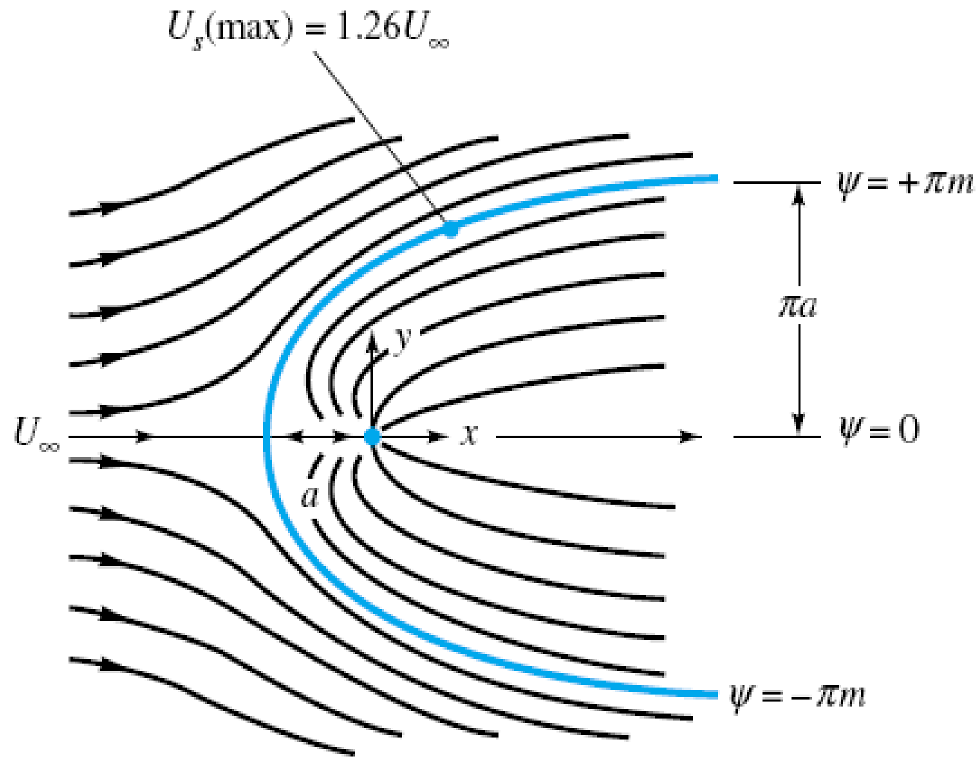
Uniform Stream Plus a Sink at the Origin: The Rankine Half-Body

If we superimpose a uniform x-directed stream against an isolated source, a half-body shape appears. If the source is at the origin, the combined stream function is, in polar coordinates,

Uniform stream plus source: $$\psi = Ur \sin\theta + m\theta \qquad (4.135)$$

'he body shape, $\psi = \pm\pi m$.

$$r = \frac{m(\pi - \theta)}{U \sin\theta}$$



Superposition of a source plus a uniform stream forms a Rankine half-body.

$$u = \frac{\partial \psi}{\partial y} = U + \frac{m}{r}\cos\theta \qquad \upsilon = -\frac{\partial \psi}{\partial x} = \frac{m}{r}\sin\theta$$

Setting $u = \upsilon = 0$, we find a single stagnation point at $\theta = 180°$ and $r = m/U$,

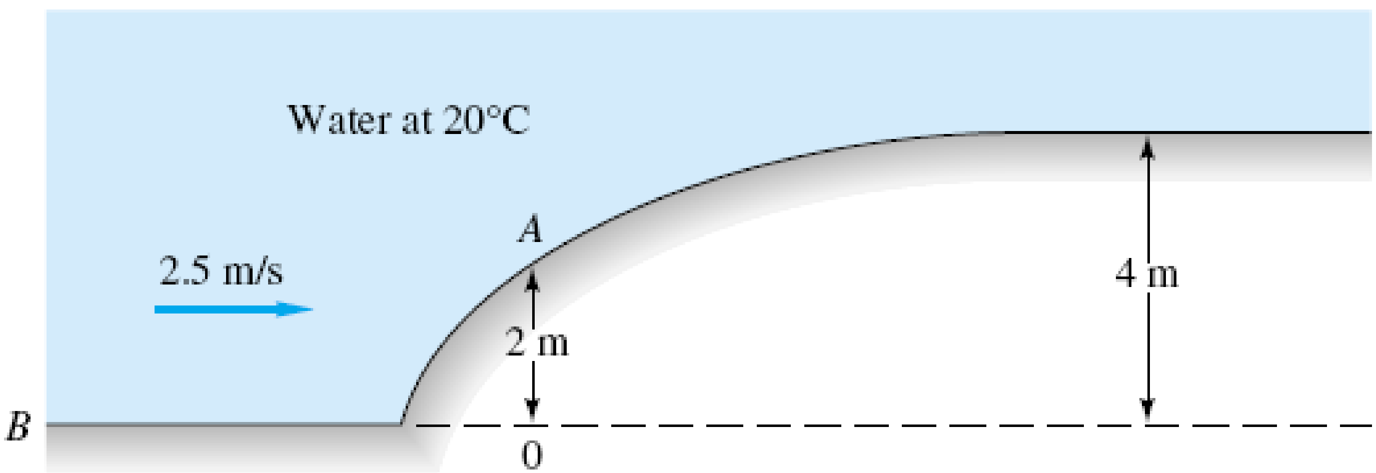
$$(x, y) = (-a, 0), \text{ where } a = m/U.$$

$$V^2 = u^2 + \upsilon^2 = U^2\left(1 + \frac{a^2}{r^2} + \frac{2a}{r}\cos\theta\right)$$

where we have substituted $m = Ua$. If we evaluate the velocities along the upper surface $\psi = \pi m$, we find a maximum value $U_{s,\max} \approx 1.26U$ at $\theta = 63°$.

EXAMPLE 4.10

The bottom of a river has a 4-m-high bump which approximates a Rankine half-body, as in Fig. E4.10. The pressure at point B on the bottom is 130 kPa, and the river velocity is 2.5 m/s. Use inviscid theory to estimate the water pressure at point A on the bump, which is 2 m above point B.



$$r = \frac{m(\pi - \theta)}{U \sin \theta} \qquad a = m/U. \qquad a = (4\text{ m})/\pi = 1.27\text{ m.}$$

$$h = 2\text{ m} = \pi a/2. \qquad \theta = \frac{\pi}{2} = 90^\circ$$

$$V_A^2 = U^2\left[1 + \frac{a^2}{(\pi a/2)^2} + \frac{2a}{\pi a/2}\cos\frac{\pi}{2}\right] = 1.405U^2$$

$$V_A \approx 1.185U = 1.185(2.5\text{ m/s}) = 2.96\text{ m/s}$$

For water at 20°C, take $\rho = 998\text{ kg/m}^2$ and $\gamma = 9790\text{ N/m}^3$.

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A \approx \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B$$

$$\frac{p_A}{9790\text{ N/m}^3} + \frac{(2.96\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 2\text{ m} \approx \frac{130{,}000}{9790} + \frac{(2.5)^2}{2(9.81)} + 0$$

$$p_A = (13.60 - 2.45)(9790) \approx 109{,}200\text{ Pa} \qquad \textit{Ans.}$$

$$u = \frac{\partial \psi}{\partial y} \qquad \upsilon = -\frac{\partial \psi}{\partial x}$$

The condition of irrotationality reduces to Laplace's equation for ψ also:

$$2\omega_z = 0 = \frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

or

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\upsilon_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \upsilon_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

8.2 Elementary Plane-Flow Solutions

Uniform stream $\mathbf{i}U$:	$\psi = Uy$	$\phi = Ux$
Line source or sink:	$\psi = m\theta$	$\phi = m \ln r$
Line vortex:	$\psi = -K \ln r$	$\phi = K\theta$

If the uniform stream is written in plane polar coordinates, it becomes

Uniform stream $\mathbf{i}U$: $\psi = Ur \sin \theta \qquad \phi = Ur \cos \theta$

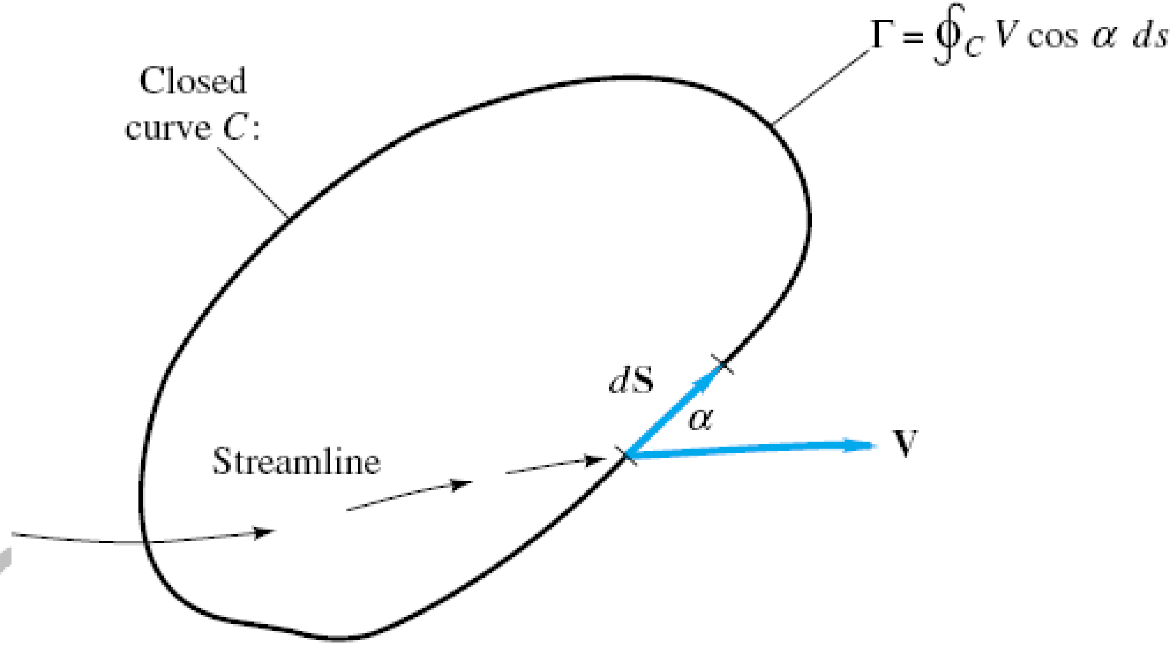
If the uniform stream is moving at angle α with respect to the x-axis, i.e.,

$$u = U \cos \alpha = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \upsilon = U \sin \alpha = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$\psi = U(y \cos \alpha - x \sin \alpha) \qquad \phi = U(x \cos \alpha + y \sin \alpha)$$

Circulation

$$\Gamma = \oint_C V \cos \alpha \, ds = \int_C \mathbf{V} \cdot d\mathbf{s} = \int_C (u\,dx + v\,dy + w\,dz)$$



From the definition of ϕ, $\mathbf{V} \cdot d\mathbf{s} = \nabla\phi \cdot d\mathbf{s} = d\phi$ for an irrotational flow;

for vortex flow: With $\phi = K\theta$ $\quad \Gamma = 2\pi K$

Alternately the calculation can be made by defining a circular path of radius r around the vortex center,

$$\Gamma = \int_C v_\theta \, ds = \int_0^{2\pi} \frac{K}{r} r \, d\theta = 2\pi K \qquad (8.17)$$

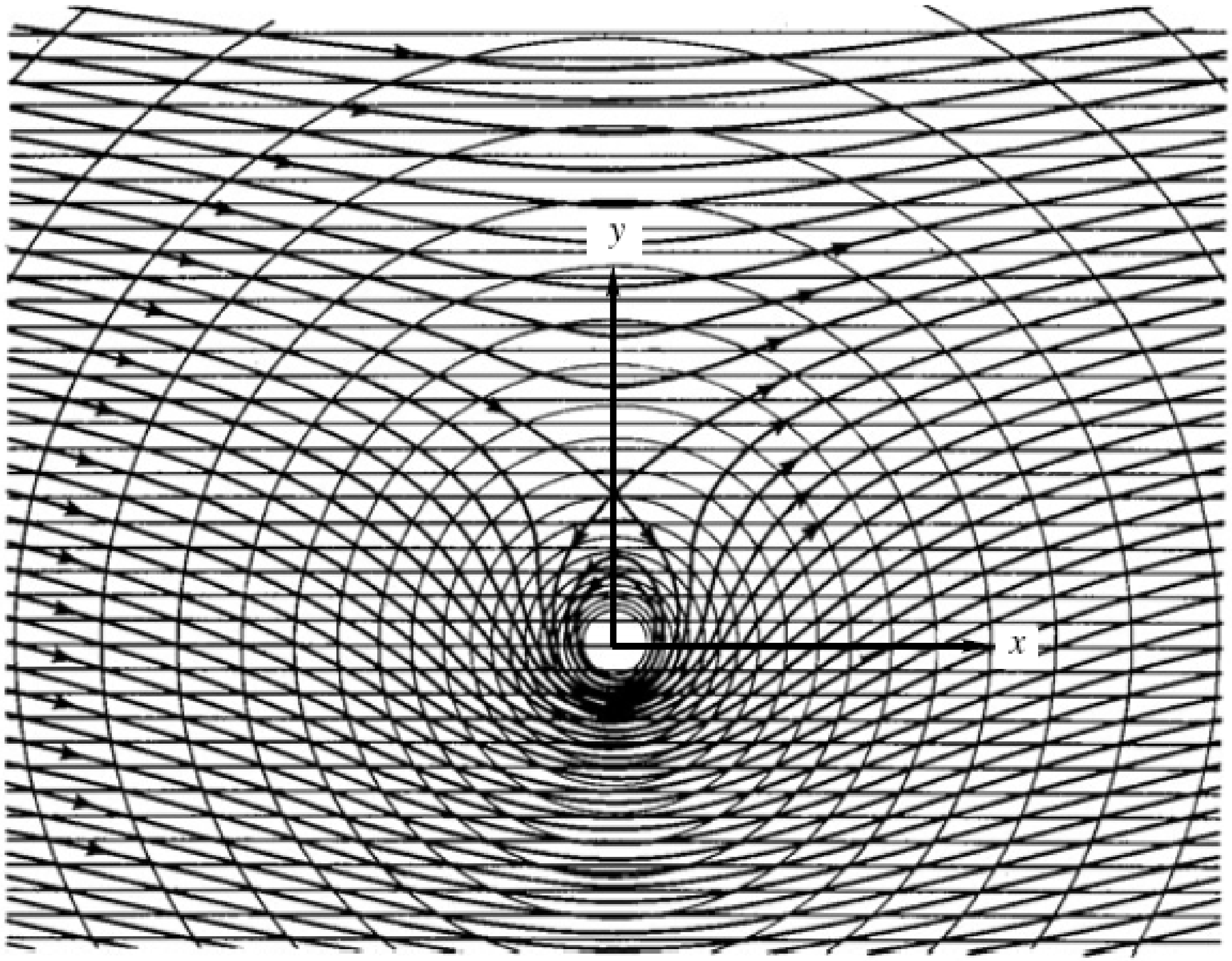
8.3 Superposition of Plane-Flow Solutions

$$\psi_{tot} = \sum \psi_i$$

Flow Past a Vortex

$$\psi = \psi_{stream} + \psi_{vortex} = U_\infty r \sin\theta - K \ln r$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty \cos\theta \qquad v_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty \sin\theta + \frac{K}{r}$$



Flow of a uniform stream past a vortex constructed by the graphical method.

An Infinite Row of Vortices

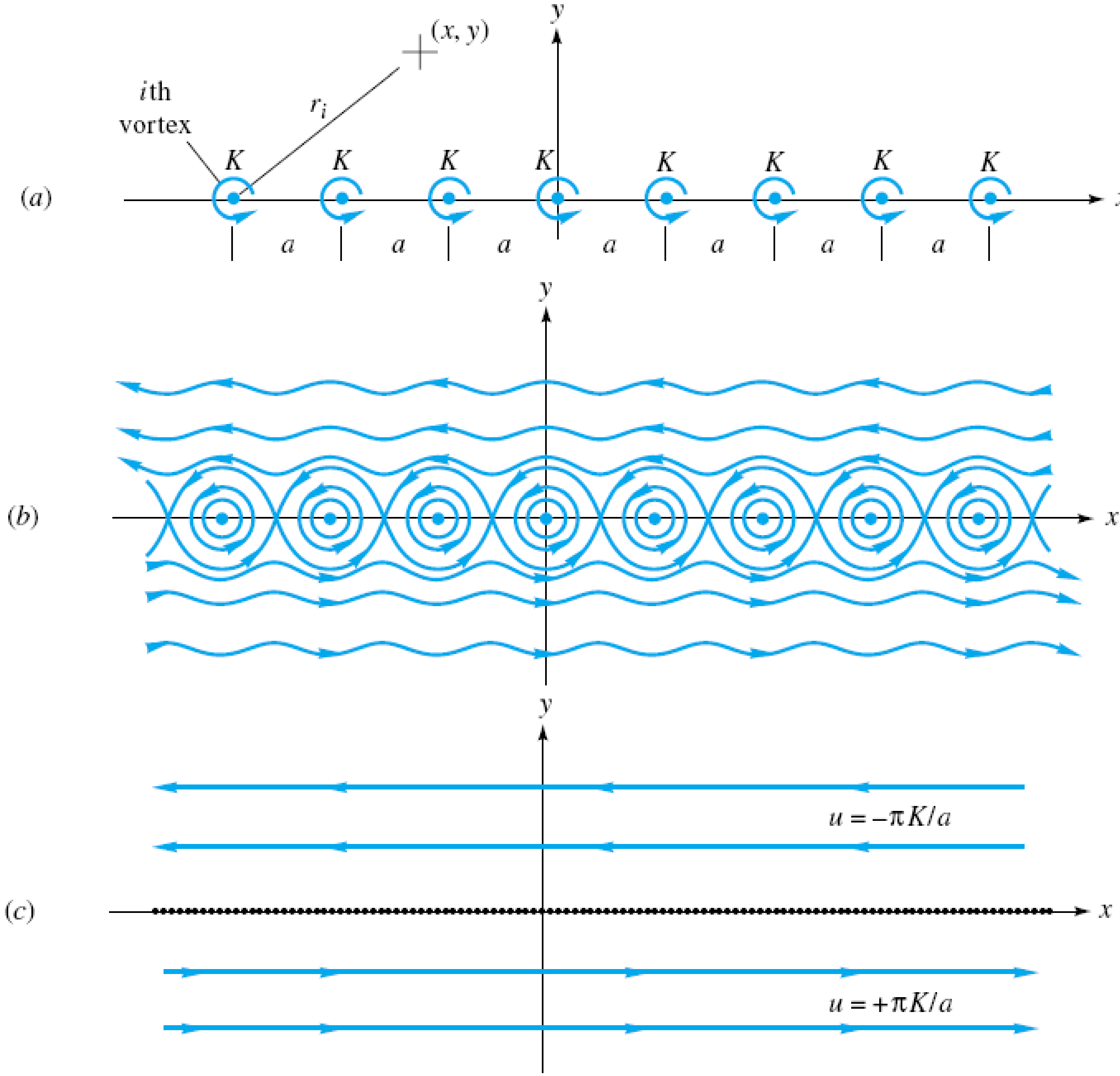
Consider an infinite row of vortices of equal strength K and equal spacing a,

a *vortex sheet*.

$$\psi_i = -K \ln r_i, \qquad \psi = -K \sum_{i=1}^{\infty} \ln r_i$$

$$\psi = -\tfrac{1}{2}K \ln\left[\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right)\right] \tag{8.22}$$

Since the proof uses the complex variable $z = x + iy$, $i = (-1)^{1/2}$, we are not going to show the details here.


(a)
(x, y)
y
ith
vortex
r_i
K
a
x
(b)
y
x
(c)
y
x
$u = -\pi K/a$
$u = +\pi K/a$

As we move far away from the source-sink pair of Fig. 4.13, the flow pattern begins to resemble a family of circles tangent to the origin, as in Fig. 8.8. This limit of vanishingly small distance a is called a *doublet*.

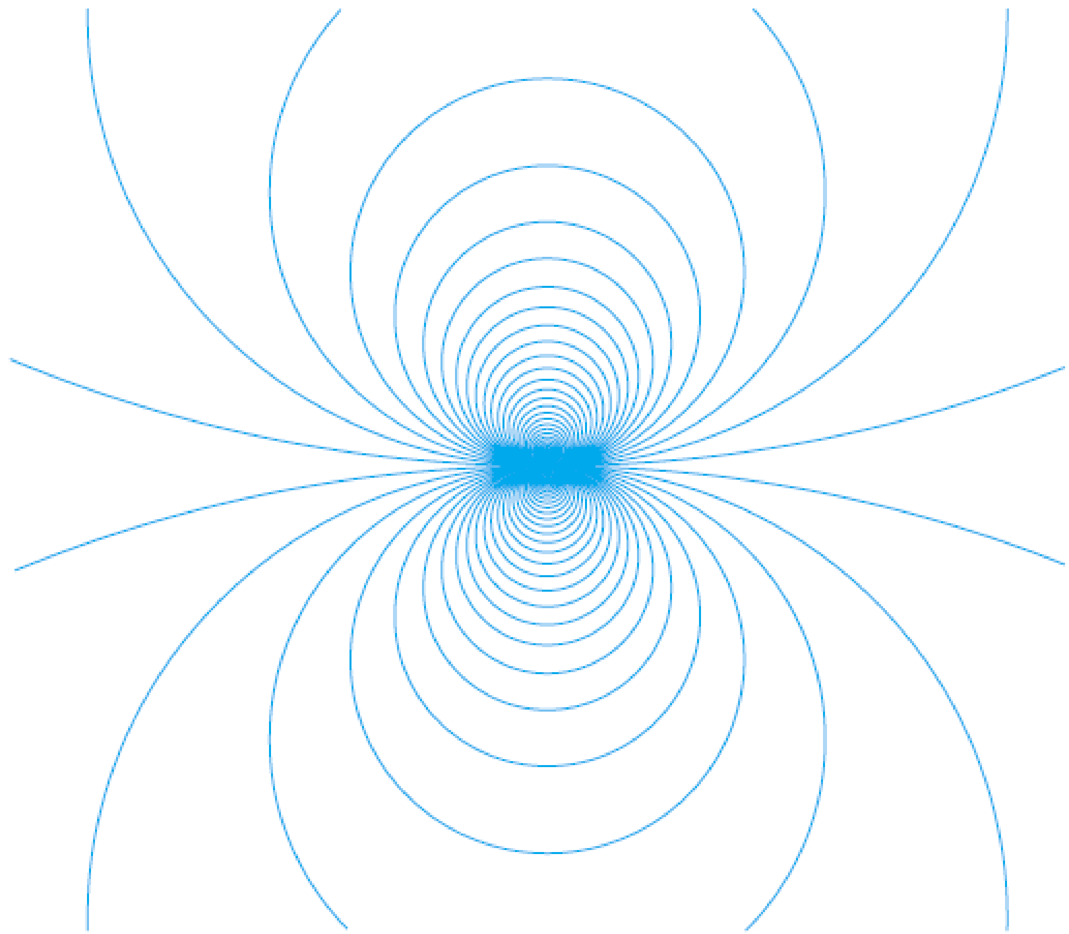


Fig. 8.8 A doublet, or source-sink pair, is the limiting case of Fig. 4.13 viewed from afar. Streamlines are circles tangent to the x-axis at the origin. This figure was prepared using the *contour* feature of MATLAB [34, 35].

product $2am$ re main constant. Let us call this constant λ.

$$\psi = \lim_{\substack{a\to 0\\ 2am=\lambda}} \left(-m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2} \right) = -\frac{2amy}{x^2 + y^2} = -\frac{\lambda y}{x^2 + y^2} \qquad (8.26)$$

$$x^2 + \left(y + \frac{\lambda}{2\psi} \right)^2 = \left(\frac{\lambda}{2\psi} \right)^2$$

$$\phi_{\text{doublet}} = \frac{\lambda x}{x^2 + y^2} \qquad \text{or} \qquad \left(x - \frac{\lambda}{2\phi} \right)^2 + y^2 = \left(\frac{\lambda}{2\phi} \right)^2$$

The doublet functions can also be written in polar coordinates

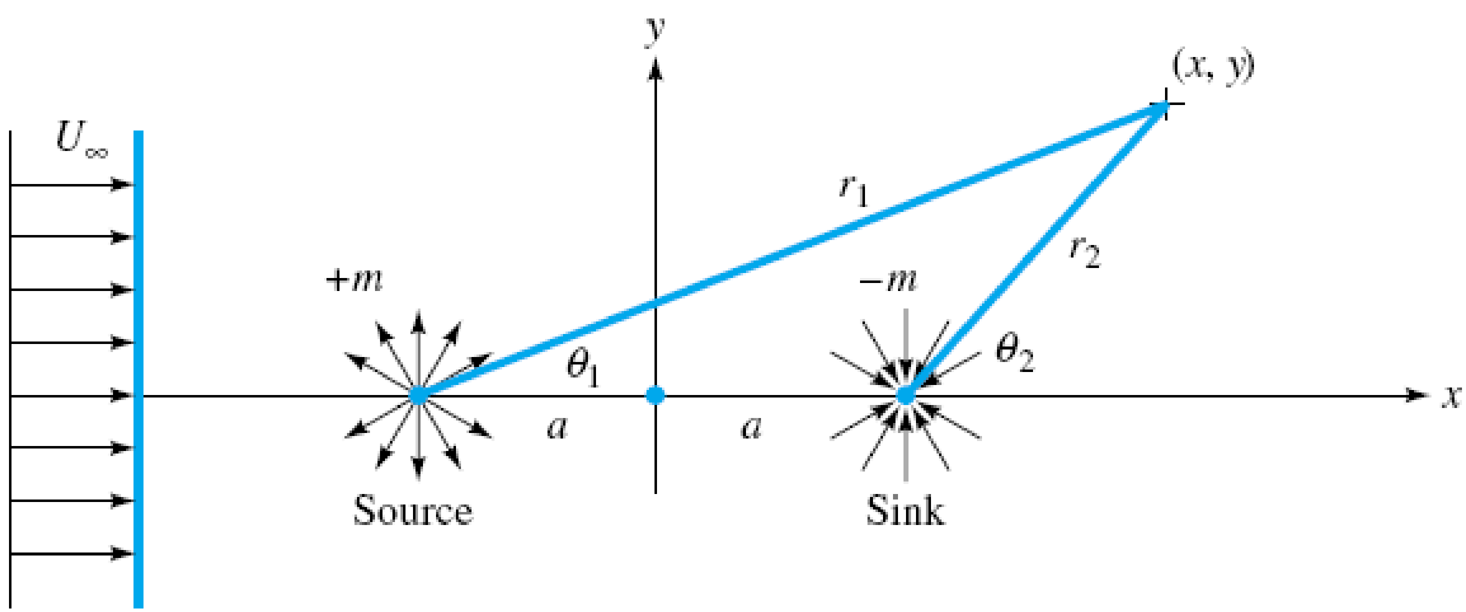
$$\psi = -\frac{\lambda \sin\theta}{r} \qquad \phi = \frac{\lambda \cos\theta}{r}$$

8.4 Plane Flow Past Closed-Body Shapes

The body shape will be closed only if the net source outflow equals the net sink inflow.

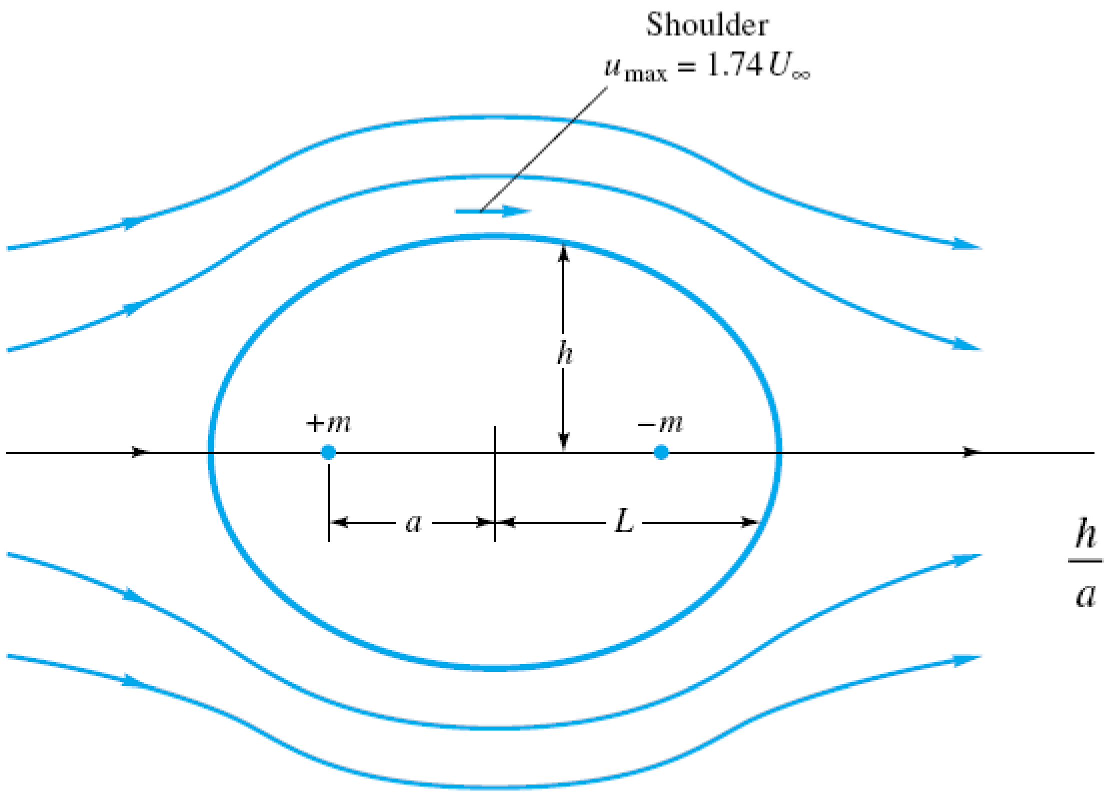
The Rankine Oval

formed by a source-sink pair aligned parallel to a uniform stream,



(a)

$$\psi = U_\infty y - m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2} = U_\infty r \sin \theta + m(\theta_1 - \theta_2) \qquad (8.29)$$



(b)

(b) oval shape and streamlines for $m/(U_\infty a) = 1.0$.

$$\frac{h}{a} = \cot \frac{h/a}{2m/(U_\infty a)} \qquad \frac{L}{a} = \left(1 + \frac{2m}{U_\infty a}\right)^{1/2}$$

$$\frac{u_{max}}{U_\infty} = 1 + \frac{2m/(U_\infty a)}{1 + h^2/a^2}$$

There are stagnation points at the front and rear, $x = \pm L$,

maximum velocity and minimum pressure at the shoulders, $y = \pm h$,