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New exact solutions for free vibrations of thin orthotropic rectangular plates

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ABSTRACT

In this paper, a novel separation of variables is presented for solving the exact solutions for the free vibrations of thin orthotropic rectangular plates with all combinations of simply supported (S) and clamped (C) boundary conditions, and the correctness of the exact solutions are proved mathematically. The exact solutions for the three cases SSCC, SCCC, and CCCC are successfully obtained for the first time, although it was believed that they are unable to be obtained. The new exact solutions are further validated by extensive numerical comparisons with the solutions of FEM and those available in the literature.

1. Introduction

The orthotropic plates are commonly used in the fields of structural engineering and considered as the fundamental structural elements [1–3] in aerospace, naval and ocean structures. The orthotropic behavior not only arises from the use of materials with such constitutive relations, many composite plates may be modeled analytically as orthotropic plates [4]. Isotropic plates altered by metallurgical process along perpendicular directions, and panels unequally stiffened along two orthogonal directions, also exhibit orthotropic characteristics [1,4]. The wide use of such structures requires investigating the vibration characteristics of orthotropic plates in order to develop accurate and reliable design. The study of the free vibration of plates dates back to the 1880s, see references [5–7], as reported in the literature survey of Liew and Xiang [8]. And an extensive review of the works up to 1985 has been given by Leissa in his monograph [9] and in a series of review articles [10–13].

Problems involving rectangular plates fall into three distinct categories [14]: (a) plates with all edges simply supported; (b) plates with a pair of opposite edges simply supported; (c) plates which do not fall into any of the above categories. Problems of the first and second categories are amenable to straightforward rigorous analysis in terms of the well-known Navier and Levy solutions [15]. These methods can be simply extended to orthotropic plates [16]. However, owing to coupled multiple differential equations of high order, it was believed that the problems of the third category are difficult to deal with ([1,4] for example), rigorous analytical solutions, which satisfy the governing differential equation and the boundary conditions exactly, turn out to be rare. For this reason many efforts were devoted to develop approximate methods [17–48].

Due to its high versatility and conceptual simplicity [17], the Rayleigh–Ritz method is one of the most popular methods to obtain approximate solutions for the natural frequencies of an orthotropic rectangular plate. Hearmon [18] proposed an approximate general solution for the free vibrations of the orthotropic plates applying the Rayleigh method with characteristic beam functions. Using a set of boundary characteristic orthogonal polynomials proposed by Bhat [19], Dickinson and Di Blasio [20] calculated the natural frequencies of rectangular orthotropic plates with various boundary conditions. Particularly interesting, among the papers using the Rayleigh–Ritz method, is the contribution of Marangoni et al. [21], wherein the Rayleigh–Ritz method and the decomposition technique presented by Bazely et al. [22] were extended to compute the upper and lower bounds of vibration frequencies for clamped orthotropic rectangular plates. Rossi et al. [2] have used the optimized Rayleigh–Ritz method and a pseudo-Fourier expansion to analyze the plates with one or more free edges, their results showed excellent agreement with those obtained by means of finite element method.

The method based on superposition of appropriate Levy type solutions for the analysis of rectangular plates was first illustrated by Timoshenko and Krieger [23]. Gorman extended this method to the free vibration analyses of isotropic [24], clamped orthotropic [25], free orthotropic [26,27], point supported orthotropic [28], and laminated symmetric cross-ply rectangular plates [29]. It has been shown [30] that the approach is powerful for such problems since the governing differential equation is satisfied rigorously at every stage and the boundary conditions can be satisfied in a series sense to any desired degree of accuracy. Yu and Cleghorn [31] em-

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ployed the superposition method and the affined transformation developed by Brunelle and Oyibo [32] to obtain vibration frequencies for orthotropic rectangular plates with combinations of clamped and simply supported edges, their results manifest high accuracy through comparisons with the upper and lower bounds of Marangoni et al. [21]. Recently, Bhaskar and Sivaram [33] expounded a novel superposition approach for the problems of static flexure, the distinguishing feature being the use of untruncated series counterparts of the conventional lengthy Levy-type expressions without any loss of accuracy. Kshirsagar and Bhaskar [34] extended this method for the free vibration and buckling studies of orthotropic rectangular plates with any combination of the conventional edge conditions.

The Kantorovich method [35] of reducing a partial differential equation to an ordinary differential was extended, respectively by Jones and Milne [36], and Bhat et al. [37] to study free vibration of isotropic rectangular plates. Dalaei and Kerr [38], and Bercin [39] used the method in reference [36] to obtain natural frequencies of fully clamped orthotropic thin plates. Sakata et al. [40] applied the method in reference [37] to the vibration analysis of rectangular orthotropic plates, obtaining very accurate results. They emphasized that the method is simpler than others available in the literature such as, for example, the Rayleigh–Ritz method that requires a larger computing effort.

Several other methods have also been investigated by researchers. Biancolini et al. [1] proposed a simplified approximate method to evaluate the natural frequencies of an orthotropic plate. They proclaimed the method is suitable to execute simple preliminary design considerations for fast final general checks of accuracy. Jayaraman et al. [41] have studied free vibrations of rectangular orthotropic plates with the two parallel sides simply supported using an exact analysis. Ramkumar et al. [42] employed the Lagrange multiplier technique to study the free vibration behavior of clamped orthotropic plates. Al-Khaiat [43] employed an initial value method to analyze the vibration of rectangular orthotropic plates. Chen [44] used an iterative approach based on finite difference equations to calculate the fundamental vibration frequency of an orthotropic plate. Huang et al. [45] have used the Green function to analyze the free vibration of orthotropic rectangular plates with variable thickness and general boundary conditions. Bardell et al. [46,47] have studied the free vibrations of specially orthotropic plates and generally orthotropic coplanar plate assemblies using a new *h*–*p* finite element methodology. Excellent agreement was found with the work of other investigators. Tsay and Reddy [48] developed a finite element model that is very convenient, especially when dealing with every-day design-type problems.

Of all the available solutions, those based on an exact approach, wherein the governing equations and the boundary conditions are satisfied rigorously, are valuable and computationally efficient [33]; there is renewed interest in such classical solutions because the solution methodologies are often applicable with minor changes to modern state-of-the-art laminated plate structures made up of functionally graded materials or those with magneto–electro–thermo–elastic coupling effects ([49,50] for example). However, it is hitherto believed that there are no exact solutions when at least two opposite sides of a rectangular plate, whether isotropic or orthotropic, are not simply supported ([1,4] for example), and many researchers have devoted to develop approximate solutions with a high level of accuracy, see references [9–13] and above review.

In this context, the objective of this work is to solve new exact solutions with reference to the title problem by using a novel separation of variables. The remainder is organized as follows. In Section 2, the closed-form formulation of natural mode is solved by using the method of separation of variables, and its correctness is proved mathematically. In Section 3, the exact normal eigenfunctions and eigenvalue equations for the boundary condition combinations SSCC, SCCC and CCCC are obtained through the mode formulation and boundary conditions. In Section 4, the solution method of the transcendental eigenvalue equations is discussed. Numerical comparison studies are presented in Section 5, the conclusion follows.

2. The closed-form natural mode

An orthotropic material is characterized by the fact that the mechanical elastic properties have two perpendicular planes of symmetry. Due to this only four elastic constants E_1, E_2, G_{12}, υ_{12} are independent. The coefficient υ_{21} can be determined according to following relation

$$\frac{\upsilon_{12}}{E_1} = \frac{\upsilon_{21}}{E_2} \tag{1}$$

Defining some orthotropic bending stiffness parameters as

$$D_1 = \frac{E_1 h^3}{12(1-\upsilon_{12}\upsilon_{21})}, \quad D_2 = \frac{E_2 h^3}{12(1-\upsilon_{12}\upsilon_{21})}, \quad D_{66} = \frac{G_{12}h^3}{12} \\ D_{12} = \upsilon_{12}D_2 = \upsilon_{21}D_1, \quad D_3 = D_{12} + 2D_{66} \tag{2}$$

where h is the thickness of the uniform plate, see Fig. 1. Using the Love–Kirchhoff's hypotheses, the differential equation of the free vibration of orthotropic thin plate has the form

$$D_1\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2D_3\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w(x,y,t)}{\partial y^4} + \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} \\ = 0 \tag{3}$$

The solution of normal harmonic vibration of the plate can be written as

$$w(x,y,t) = W(x,y)(A\cos\omega t + B\sin\omega t) \tag{4}$$

Substitution of Eq. (4) into Eq. (3) results in the following partial differential equation involved the natural mode $W(x,y)$ as

$$D_1\frac{\partial^4 W}{\partial x^4} + 2D_3\frac{\partial^4 W}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 W}{\partial y^4} - \beta^4 W = 0 \tag{5}$$

where the frequency parameter $\beta^4 = \omega^2\rho h$. The natural mode function $W(x, y)$ can be solved from Eq. (5) in conjunction with the boundary conditions. The simple or classical boundary conditions for thin plate include the free, the simply supported and the clamped. Consider a separation of variable solution

$$W(x,y) = \phi(x)\psi(y) \tag{6}$$

to the homogeneous governing Eq. (5), the following equation can be obtained

$$D_1\phi^{(\mathrm{IV})}\psi + 2D_3\phi''\psi'' + D_2\phi\psi^{(\mathrm{IV})} - \beta^4\phi\psi = 0 \tag{7}$$

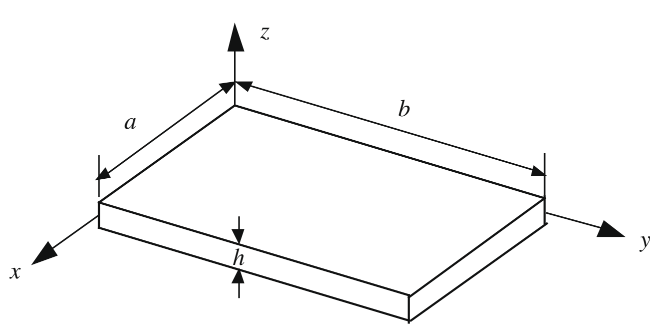


Fig. 1. A rectangular plate and coordinates.

In order for the separation of variables to occur, it was requested that

$$\psi'' = -\gamma^2\psi \quad \text{or} \quad \phi'' = -\alpha^2\phi \tag{8}$$

Thus two opposite edges of the plates must be simply supported, i.e. $\psi = \sin\gamma y$ or $\varphi = \sin\alpha x$, so that the condition (8) can be satisfied.In present paper, the authors assume the eigenfunctions in Eq. (7) as

$$\phi(x) = Ae^{\mu x}, \quad \psi(y) = Be^{\lambda y} \tag{9}$$

where the variables μ and λ are the eigenvalues corresponding to the eigenfunctions $\varphi(x)$ and $\psi(y)$, respectively. Substituting Eq. (9) into Eq.(7), one can obtain

$$D_1\mu^4 + 2D_3\mu^2\lambda^2 + D_2\lambda^4 - \beta^4 = 0 \tag{10}$$

This is the characteristic equation of Eq. (7) or Eq. (5). By solving Eq. (10), one can obtain the characteristic roots

$$\mu_{1,2} = \pm i\sqrt{\vartheta_1 + \delta_1} \triangleq \pm i\alpha_1, \quad \mu_{3,4} = \pm\sqrt{\vartheta_1 - \delta_1} \triangleq \pm\beta_1 \tag{11ab}$$

where $i^2 = -1$, and

$$\delta_1 = \lambda^2\frac{D_3}{D_1}, \quad \vartheta_1 = \sqrt{\lambda^4\left[\left(\frac{D_3}{D_1}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\beta^4}{D_1}} \tag{12}$$

In Eq. (11), μ is expressed by λ. Conversely, λ can be expressed by μ, that is

$$\lambda_{1,2} = \pm i\sqrt{\vartheta_2 + \delta_2} \triangleq \pm i\alpha_2, \quad \lambda_{3,4} = \pm\sqrt{\vartheta_2 - \delta_2} \triangleq \pm\beta_2 \tag{13ab}$$

where

$$\delta_2 = \mu^2\frac{D_3}{D_2}, \quad \vartheta_2 = \sqrt{\mu^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\beta^4}{D_2}} \tag{14}$$

Substitution of $\mu = i\alpha_1$, see Eq. (11a), into Eq. (13) leads to

$$\alpha_2 = \sqrt{\sqrt{\alpha_1^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\beta^4}{D_2}} - \alpha_1^2\frac{D_3}{D_2}} \tag{15a}$$

$$\beta_2 = \sqrt{\sqrt{\alpha_1^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\beta^4}{D_2}} + \alpha_1^2\frac{D_3}{D_2}} \tag{15b}$$

Eliminating λ from Eq. (11), one can have

$$(\alpha_1^2 + \beta_1^2)^2 + \frac{D_1D_2 - D_3^2}{D_3^2}(\alpha_1^2 - \beta_1^2)^2 = \frac{4\beta^4}{D_1} \tag{16}$$

It is noteworthy that Eqs. (15) and (16) will be used to solve the frequencies, this is done below. According to the characteristic roots in Eqs. (11) and (13), the two eigenfunctions or the two Levy's solutions in Eq. (9) can be written as

$$\phi(x) = A_1\cos\alpha_1x + B_1\sin\alpha_1x + C_1\cosh\beta_1x + H_1\sinh\beta_1x \tag{17}$$

$$\psi(y) = A_2\cos\alpha_2y + B_2\sin\alpha_2y + C_2\cosh\beta_2y + H_2\sinh\beta_2y \tag{18}$$

The remaining problem is to prove the solution $W(x,y) = \varphi(x)\psi(y)$ in Eq. (6), where $\varphi(x)$ and $\psi(y)$ are given in Eqs. (17) and (18), is the general solution of Eq. (5), and the prove is presented in Appendix A.

For rectangular thin plate, as shown in Fig. 1, there are two boundary conditions at each side, so only one of the eight integral constants in Eqs. (17) and (18) is independent, and the integral constants and both eigenvalue equations can be derived exactly by means of the eight boundary conditions. It follows from Eqs. (15) and (16) that if α_1, β_1 and the frequency parameter β have been solved, then the eigenvalues α_2 and β_2 can be solved accordingly, that means α_1, β_1 and β can be considered as independent parameters. The solutions methods for eigenvalues and frequencies will be presented below.

3. Eigenvalue equations and eigenfunctions

Regardless of the two opposite edges being S–S, or S–C or C–C, as shown in Fig. 2, the exact solutions of eigenfunctions and eigenvalue equations can be solved similarly, therefore only the case C–C is solved to show the solution methods of eigenfunctions and eigenvalues. Assume both edges $x = 0$ and $x = a$ are clamped (i.e. the case C–C), the boundary conditions have the form

$$\begin{aligned} &W(0,y) = 0 \Rightarrow \phi(0) = 0, \quad W(a,y) = 0 \Rightarrow \phi(a) = 0 \\ &\partial W(0,y)/\partial x = 0 \Rightarrow \phi'(0) = 0 \quad \partial W(a,y)/\partial x = 0 \Rightarrow \phi'(a) = 0 \end{aligned} \tag{19}$$

Substitution of Eq. (17) into Eq. (19) results in four homogeneous algebraic equations for unknown constants A_1, B_1, C_1 and H_1

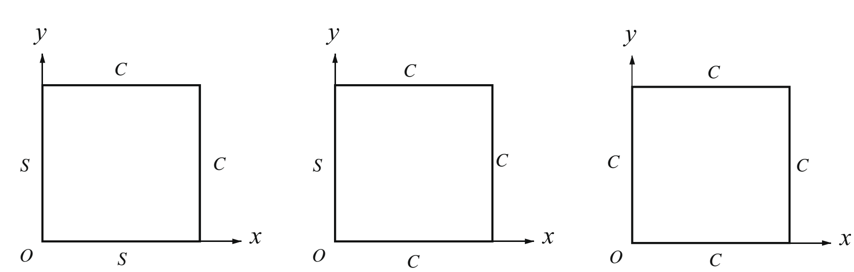
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha_1 & 0 & \beta_1 \\ \cos\alpha_1a & \sin\alpha_1a & \cosh\beta_1a & \sinh\beta_1a \\ -\alpha_1\sin\alpha_1a & \alpha_1\cos\alpha_1a & \beta_1\sinh\beta_1a & \beta_1\cosh\beta_1a \end{pmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20abcd}$$

It can be solved from Eqs. (20a) and (20b) that

$$\begin{aligned} &A_1 = -C_1 \\ &\alpha_1B_1 = -\beta_1H_1 \end{aligned} \tag{21}$$

Then Eq. (20) can be rewritten as

$$\begin{pmatrix} \cos\alpha_1a - \cosh\beta_1a & \frac{\beta_1}{\alpha_1}\sin\alpha_1a - \sinh\beta_1a \\ \alpha_1\sin\alpha_1a + \beta_1\sinh\beta_1a & \beta_1(-\cos\alpha_1a + \cosh\beta_1a) \end{pmatrix} \begin{bmatrix} C_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22ab}$$



(a) SSCC (b) SCCC (c) CCCC

Fig. 2. The boundary conditions of plate.

Table 1
The eigensolutions for the cases SSCC, SCCC and CCCC.

	Eigenvalue equations	Normal eigenfunctions
SSCC	$\frac{\tan\alpha_1 a}{\alpha_1 a}-\frac{\tanh\beta_1 a}{\beta_1 a}=0$	$\phi(x)=\sin\alpha_1 x-\frac{\sin\alpha_1 a}{\sinh\beta_1 a}\sinh\beta_1 x$
	$\frac{\tan\alpha_2 b}{\alpha_2 b}-\frac{\tanh\beta_2 b}{\beta_2 b}=0$	$\psi(y)=\sin\alpha_2 y-\frac{\sin\alpha_2 b}{\sinh\beta_2 b}\sinh\beta_2 y$
SCCC	$\frac{\tan\alpha_1 a}{\alpha_1 a}-\frac{\tanh\beta_1 a}{\beta_1 a}=0$	$\phi(x)=\sin\alpha_1 x-\frac{\sin\alpha_1 a}{\sinh\beta_1 a}\sinh\beta_1 x$
	$\frac{1-\cos\alpha_2 b\cosh\beta_2 b}{\sin\alpha_2 b\sinh\beta_2 b}=\frac{\alpha_2^2-\beta_2^2}{2\alpha_2\beta_2}$	$\psi(y)=-\cos\alpha_2 y+(\beta_2/\alpha_2)k_1\sin\alpha_2 y+\cosh\beta_2 y-k_1\sinh\beta_2 y$ $k_1=\frac{\cos\alpha_2 a-\cosh\beta_2 a}{(\beta_2/\alpha_2)\sin\alpha_2 a-\sinh\beta_2 a}$
CCCC	$\frac{1-\cos\alpha_1 a\cosh\beta_1 a}{\sin\alpha_1 a\sinh\beta_1 a}=\frac{\alpha_1^2-\beta_1^2}{2\alpha_1\beta_1}$	$\phi(x)=-\cos\alpha_1 x+(\beta_1/\alpha_1)k_2\sin\alpha_1 x+\cosh\beta_1 x-k_2\sinh\beta_1 x$
	$\frac{1-\cos\alpha_2 b\cosh\beta_2 b}{\sin\alpha_2 b\sinh\beta_2 b}=\frac{\alpha_2^2-\beta_2^2}{2\alpha_2\beta_2}$	$\psi(y)=-\cos\alpha_2 y+(\beta_2/\alpha_2)k_1\sin\alpha_2 y+\cosh\beta_2 y-k_1\sinh\beta_2 y$

For obtaining nontrivial solutions, the determinant of the coefficients matrix of the above homogeneous Eq. (22) must be zero, thus the eigenvalue equation can be obtained as

$$\frac{1-\cos\alpha_1 a\cosh\beta_1 a}{\sin\alpha_1 a\sinh\beta_1 a}=\frac{\alpha_1^2-\beta_1^2}{2\alpha_1\beta_1} \tag{23}$$

And one can also obtain from Eq. (22a) that

$$H_1=-k_2C_1 \quad \text{where} \quad k_2=\frac{\cos\alpha_1 a-\cosh\beta_1 a}{(\beta_1/\alpha_1)\sin\alpha_1 a-\sinh\beta_1 a} \tag{24}$$

Then the normal eigenfunction $\phi(x)$ can be obtained by substituting Eqs. (21) and (24) into Eq. (17) and assuming $C_1 = 1$ as follows:

$$\phi(x)=-\cos\alpha_1 x+(\beta_1/\alpha_1)k_2\sin\alpha_1 x+\cosh\beta_1 x-k_2\sinh\beta_1 x \tag{25}$$

The exact eigenfunctions and eigenvalue equations corresponding to the other two opposite edges $y = 0$ and $y = b$ can be obtained in the same way as above. The exact eigensolutions for cases SSCC, SCCC and CCCC as shown in Fig. 2 are presented in Table 1. It should be pointed out that the exact solutions for the three cases were not available.

4. Solution method of eigenvalue equations

It is apparent from Table 1 that there are five quantities α_1, β_1, β, α_2 and β_2 in any two eigenvalue equations, but only three of them are independent, for example α_1, β_1 and β, here α_2 and β_2 are calculated by using Eqs. (15a) and (15b), respectively.

The two eigenvalue equations involved α_1, β_1 and β can be solved in conjunction with Eq. (16). In this paper Newton's method is chosen to solve the transcendental equations. The initial values of Newton's method can be appropriately determined according to the characters of the eigenvalue equations. Let us take the case SSCC as an example to show the method of determining the initial values. From the definition of tangent functions in the two eigenvalue equations, it is readily to determine the intervals of $\alpha_1 a$ and $\alpha_2 b$, that is

$$\begin{aligned}&\alpha_1 a\in(m\pi, m\pi+0.5\pi),\quad m=1,2,\ldots\\&\alpha_2 b\in(n\pi, n\pi+0.5\pi),\quad n=1,2,\ldots\end{aligned} \tag{26}$$

The interval of β_1 is determined as follows. It can be solved from Eq. (11) that

$$\beta_1=\sqrt{\alpha_1^2-2\lambda^2\frac{D_3}{D_1}} \tag{27}$$

Table 2
Four types of material properties.

	Material	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	v_{12}	ρ (kg/m)
M1	T-graphite/epoxy	185	10.5	7.3	0.28	1600
M2	B-boron/epoxy	208	18.9	5.7	0.23	2000
M3	K-aryl/epoxy	76	5.6	2.3	0.34	1460
M4	E-glass/epoxy	60.7	24.8	12.0	0.23	

Table 3
Frequency parameter $\gamma=a\sqrt[4]{\omega^2\rho h/D_1}$ for plates with $a\times b$=1m $\times$ 1.2m and M1.

B.C.	Method	Mode sequence number									
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
SSCC	Exact	4.02	4.38	5.09	6.06	7.12	7.17	7.27	7.59	8.09	8.34
	FEM	4.02	4.39	5.10	6.07	7.12	7.17	7.28	7.59	8.09	8.35
SCCC	Exact	4.04	4.49	5.28	6.30	7.12	7.30	7.44	7.66	8.21	8.63
	FEM	4.05	4.50	5.29	6.31	7.12	7.31	7.45	7.66	8.21	8.64
CCCC	Exact	4.80	5.08	5.68	6.56	7.60	7.89	8.03	8.31	8.74	8.76
	FEM	4.81	5.10	5.70	6.57	7.62	7.90	8.04	8.32	8.75	8.77

Table 4
Frequency parameter $\gamma=a\sqrt[4]{\omega^2\rho h/D_1}$ for CCCC plates with variable b and M2.

b	Method	Mode sequence number									
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	Exact	4.87	5.50	6.68	7.91	8.15	8.16	8.72	9.62	9.75	10.81
	FEM	4.88	5.52	6.70	7.92	8.17	8.18	8.73	9.64	9.77	10.83
2	Exact	4.75	4.82	5.00	5.32	5.78	6.37	7.05	7.79	7.85	7.90
	FEM	4.75	4.83	5.01	5.33	5.80	6.39	7.06	7.80	7.87	7.91
3	Exact	4.74	4.76	4.81	4.90	5.05	5.26	5.54	5.89	6.28	6.71
	FEM	4.74	4.76	4.81	4.91	5.06	5.27	5.55	5.89	6.29	6.72

Substituting $\lambda = i\alpha_2$ into Eq. (27) leads to

$$\beta_1 = \sqrt{2\alpha_2^2 \frac{D_3}{D_1} + \alpha_1^2} \tag{28}$$

Thus the interval of β_1 is determined by Eq. (28), wherein the intervals of α_1 and α_2 are determined by Eq. (26). The middle values of the intervals in Eqs. (26) are recommended as the initial values of $\alpha_1 a$ and $\alpha_2 b$, and the initial value of β_1 can be computed through Eq. (28).

5. Numerical comparisons and discussion

The main purpose of present work is to solve some new exact solutions for the free vibrations of rectangular orthotropic thin plates. Although the correctness of the new exact solutions have

Table 5
Frequency parameters $\gamma = a\sqrt[4]{\omega^2 \rho h / D_1}$ for CCCC plates with variable b and M3.

b	Method	Mode sequence number									
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	Exact	4.85	5.41	6.48	7.87	7.92	8.15	8.66	9.33	9.47	10.54
	FEM	4.87	5.44	6.50	7.85	7.92	8.16	8.67	9.35	9.48	10.56
2	Exact	4.75	4.82	4.98	5.26	5.68	6.21	6.83	7.50	7.87	7.90
	FEM	4.75	4.82	4.99	5.27	5.69	6.22	6.84	7.51	7.88	7.91
3	Exact	4.74	4.76	4.81	4.90	5.03	5.22	5.47	5.77	6.12	6.52
	FEM	4.74	4.76	4.81	4.90	5.04	5.23	5.48	5.78	6.13	6.53

Table 6
Frequency parameters $\gamma_{ij} = \omega_{ij} a^2 \sqrt{\rho h / D_1}$ for $D_3 = D_1$, $D_2 = D_1$.

B.C.	b/a	References	Mode shape					
			(1, 1)	(2, 1)	(3, 1)	(1, 2)	(2, 2)	(3, 2)
SSCC	0.5	Exact	70.877	100.436	151.576	209.302	238.135	287.227
		Ref. [4]	71.081	100.803	151.906	209.377	238.347	287.542
	1.0	Exact	26.867	60.549	114.568	60.549	92.665	145.786
		Ref. [4]	27.059	60.667	114.633	60.667	92.844	145.937
	2.0	Exact	17.719	52.346	106.640	25.109	59.534	113.856
		Ref. [4]	17.770	52.343	106.649	25.201	59.587	113.894
CSCC	0.5	Exact	72.899	107.469	164.387	210.362	242.197	295.698
		Ref. [4]	73.405	108.236	165.023	210.526	242.667	296.366
	1.0	Exact	31.438	70.877	130.240	63.053	100.436	159.198
		Ref. [4]	31.833	71.081	130.353	63.340	100.803	159.487
	2.0	Exact	24.066	63.714	123.066	30.071	70.052	129.641
		Ref. [4]	24.144	63.742	123.081	30.253	70.143	129.693
CCCC	0.5	Exact	97.542	125.751	177.613	255.678	283.509	331.850
		Ref. [4]	98.324	127.333	179.115	255.939	284.325	333.125
	1.0	Exact	35.112	72.899	131.629	72.899	107.469	164.387
		Ref. [4]	35.999	73.405	131.902	73.405	108.236	165.023
	2.0	Exact	24.358	63.920	123.217	31.438	70.877	130.240
		Ref. [4]	24.581	63.985	123.249	31.833	71.081	130.353

Table 7
Frequency parameters $\gamma_{ij} = \omega_{ij} a^2 \sqrt{\rho h / D_1}$ for $D_3 = 0.5\ D_1$, $D_2 = D_1$.

	b/a	References	Mode shape					
			(1, 1)	(2, 1)	(3, 1)	(1, 2)	(2, 2)	(3, 2)
SSCC	0.5	Exact	67.331	90.528	137.293	204.990	222.753	258.282
		Ref. [4]	67.497	90.838	137.574	205.045	222.923	258.547
	1.0	Exact	24.449	56.603	110.137	56.603	82.431	131.639
		Ref. [4]	24.610	56.700	110.189	56.700	82.584	131.766
	2.0	Exact	16.833	51.248	105.504	22.632	55.688	109.558
		Ref. [4]	16.874	51.261	105.510	22.710	55.731	109.581
CSCC	0.5	Exact	69.254	97.795	150.754	205.859	226.410	266.561
		Ref. [4]	69.687	98.440	151.290	205.994	226.800	267.127
	1.0	Exact	29.296	67.331	126.178	59.021	90.528	145.751
		Ref. [4]	29.625	67.497	126.268	59.270	90.838	145.990
	2.0	Exact	23.385	62.772	122.034	27.906	66.600	125.690
		Ref. [4]	23.447	62.794	122.045	28.057	66.672	125.730
CCCC	0.5	Exact	94.725	117.182	164.294	251.755	269.323	304.619
		Ref. [4]	95.391	118.502	165.583	251.965	269.987	305.677
	1.0	Exact	33.174	69.254	127.382	69.254	97.795	150.754
		Ref. [4]	33.917	69.687	127.613	69.687	98.440	151.290
	2.0	Exact	23.681	62.939	122.150	29.296	67.331	126.178
		Ref. [4]	23.848	62.991	122.175	29.625	67.497	126.268

Table 8
Frequency parameters $\gamma_{ij} = \omega_{ij} a^2 \sqrt{\rho h / D_1}$ for $D_3 = 0.5\ D_1$, $D_2 = 0.5\ D_1$.

	b/a	References	Mode shape					
			(1, 1)	(2, 1)	(3, 1)	(1, 2)	(2, 2)	(3, 2)
SSCC	0.5	Exact	51.302	79.310	130.072	148.490	172.181	216.154
		Ref. [4]	51.507	79.602	130.294	148.564	172.386	216.429
	1.0	Exact	21.898	55.508	109.529	44.228	74.462	126.736
		Ref. [4]	22.042	55.578	109.564	44.342	74.599	126.835
	2.0	Exact	16.609	51.159	105.451	20.844	54.959	109.162
		Ref. [4]	16.638	51.168	105.455	20.910	54.990	109.178
CSCC	0.5	Exact	53.831	87.591	144.279	149.691	176.911	226.031
		Ref. [4]	54.344	88.169	144.689	149.870	177.369	226.602
	1.0	Exact	27.258	66.436	125.658	47.305	83.374	141.366
		Ref. [4]	27.527	66.552	125.719	47.584	83.640	141.547
	2.0	Exact	23.235	62.702	121.990	26.501	66.000	125.349
		Ref. [4]	23.277	62.716	121.997	26.620	66.052	125.377
CCCC	0.5	Exact	70.524	98.828	151.822	181.529	205.209	249.726
		Ref. [4]	71.371	100.126	152.844	181.816	206.026	250.861
	1.0	Exact	29.329	67.509	126.377	53.831	87.591	144.279
		Ref. [4]	29.986	67.802	126.522	54.344	88.169	144.689
	2.0	Exact	23.399	62.806	122.065	27.258	66.436	125.658
		Ref. [4]	23.504	62.840	122.082	27.527	66.552	125.719

Table 9
Frequency parameters $\gamma = a\sqrt[4]{12\rho\omega^2 / E_2 h^2}$ for orthotropic rectangular plates with M4.

	b/a	References	Mode sequence number					
			1st	2nd	3rd	4th	5th	6th
SSCC	1.0	Exact	5.803	8.087	9.339	10.703	10.911	12.806
		Ref. [45]	5.818	8.090	9.330	10.695	10.879	12.772
	2.0	Exact	5.113	5.679	6.618	7.813	8.990	9.299
		Ref. [45]	5.115	5.684	6.612	7.759	8.977	9.287
CSCC	1.0	Exact	6.119	8.676	9.437	11.007	11.599	13.189
		Ref. [45]	6.156	8.683	9.435	11.007	11.555	13.135
	2.0	Exact	5.149	5.803	6.831	8.087	9.000	9.339
		Ref. [45]	5.156	5.816	6.826	8.018	8.988	9.330
CCCC	1.0	Exact	6.714	8.921	10.297	11.605	11.720	13.663
		Ref. [45]	6.780	8.953	10.293	11.615	11.686	13.636
	2.0	Exact	6.073	6.508	7.308	8.401	9.678	9.962
		Ref. [45]	6.080	6.532	7.320	8.347	9.698	9.941

been proved mathematically (see Appendix A), extensive numerical comparison studies are also presented in this section. We focus only on the three newly solved cases SSCC, SCCC and CCCC, and the exact frequencies are compared with the results calculated by FEM and other approximate methods [4,21,31,45]. Four distinct types of materials in Table 2 and the thickness $h = 0.02$ m are used in the numerical comparison.

In Tables 3–5, the exact results are compared with the results calculated using MSC/NASTRAN with the Bending Panel element whose size is 1 cm $\times$ 1 cm. In Table 3, the exact results for cases SSCC, SCCC and CCCC are studied comparatively for M1 (see Table 2) and $a \times b = 1$ m $\times$ 1.2 m. In Table 4 and Table 5, the exact results for CCCC orthotropic plates with variable length b are compared for M2 and M3, respectively. It is found that all exact results in Tables 3–5 agree excellently with the FEM results.

In Tables 6–8, the results in reference [4], calculated through Kantorovich method, are used for comparison, and the frequency parameter γ_{ij}, corresponding to the mode shape having i and j half waves in x and y directions, respectively, are calculated for different aspect ratios b/a, different flexural rigidity ratios D_3/D_1 and D_2/D_1. It is noteworthy that the exact frequencies in Tables 3–8 are slightly smaller than the results used for comparisons, since the frequencies obtained by using the two approximate methods are the upper bounds.

In Table 9, the results [45] of Green function method are used for comparison studies which are carried out for SSCC, CSCC and

Table 10
Frequency parameters $\gamma = \omega a^2 \sqrt{\rho h / D_1}$ for CCCC orthotropic rectangular plates with $D_1 = 1.543\ D_3$, $D_2 = 4.810\ D_3$.

a/b	Mode	Upper bounds [21]	Lower bounds [21]	Ref. [31]	Exact
0.5	1	25.425	25.420	25.424	25.104
	2	37.715	37.303	37.719	37.330
1.0	1	47.482	47.473	47.481	46.741
	2	78.015	–	78.021	77.300
1.5	1	93.981	93.960	93.980	93.378
	2	115.45	–	115.47	114.33
2.0	1	161.95	161.85	161.95	161.51
	2	177.91	–	177.94	176.78

Table 11
Frequency parameters $\gamma = \omega a^2 \sqrt{\rho h / D_1}$ for CCCC orthotropic rectangular plates with $D_1 = 4.310\ D_3$, $D_2 = 0.305\ D_3$.

a/b	Mode	Upper bounds [21]	Lower bounds [21]	Ref. [31]	Exact
0.5	1	22.780	22.723	22.780	22.757
	2	24.083	23.774	24.089	24.009
1.0	1	24.566	24.488	24.564	24.358
	2	32.007	31.210	32.023	31.624
1.5	1	28.871	28.783	28.869	28.289
	2	49.230	48.243	49.354	48.825
2.0	1	36.620	36.337	36.618	35.735
	2	73.344	–	73.353	72.827

Table 12
Frequency parameters $\gamma = \omega a^2 \sqrt{\rho h / D_1}$ for CCCC orthotropic rectangular plates with $D_1 = 2.0\ D_3$, $D_2 = 1.0\ D_3$.

a/b	Mode	Upper bounds [21]	Lower bounds [21]	Ref. [31]	Exact
0.5	1	23.504	23.448	23.503	23.398
	2	27.513	26.990	27.524	27.258
1.0	1	29.981	29.894	29.979	29.329
	2	54.328	53.318	54.337	53.831
1.5	1	45.830	45.738	45.828	44.898
	2	79.423	–	79.437	78.625
2.0	1	71.365	71.269	71.362	70.524
	2	100.08	–	100.11	98.828

CCCC plates for two aspect ratios. It is apparent that the exact results are slightly larger or smaller than the results in reference [45], since in which the frequencies were obtained using Green function method in conjunction with numerical integration based on interpolation method.

In Tables 10–12, more exact results are presented for clamped orthotropic plates for different aspect ratios and different rigidity ratios D_1/D_3 and D_2/D_3. Some results used for comparison are from reference [21] wherein the Rayleigh–Ritz technique using clamped beam eigenfunctions and the decomposition technique (see reference [22]) were employed to estimate the upper bounds and lower bounds, respectively; and some results used for comparison are from reference [31] where the superposition method and the affined transformation (see reference [32]) were employed to obtain accurate natural frequencies for orthotropic rectangular plates.

It follows from Tables 10–12 that the exact results are slightly smaller than the upper bounds; all the second frequencies are within the upper and lower bounds whenever the lower bounds are available; however, for the first frequencies, except for the one of Table 11 for $a/b = 0.5$, all of them are smaller than the lower bounds, this is bound to the inaccuracy of the lower bounds. Marangoni et al. [21] have stated that their lower bounds are less accurate than the upper bounds. And the upper bounds and the lower bounds for the first modes are almost the same.

All comparisons in Tables 3–12 are limited to the three cases SSCC, SCCC, CCCC, as shown in Fig. 2, the exact solutions of which are obtained for the first time. All the exact results agree perfectly with the results used for comparisons, therefore the present exact solutions are validated.

6. Conclusions

For the free vibrations of rectangular orthotropic plates, the general mathematical expression of natural mode has been derived here by means of the novel separation of variables. The general solution satisfies the governing equation of the eigenvalue problem exactly, and is applicable for all kinds of boundary conditions. As would be clear to anyone familiar with analysis of plates/shells, the present method can be directly extended for buckling analysis of plates.

In present study, the exact mode functions and the frequency equations for the cases SSCC, SCCC and CCCC were obtained for the first time, since no such results have been reported heretofore. It is expected that the new exact results will provide other researchers with data against which they can compare their results.

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Appendix A

Substituting $\mu = \beta_1$ into Eq. (13), one can have

$$\alpha_2 = \sqrt{\sqrt{\beta_1^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\beta^4}{D_2}} + \beta_1^2\frac{D_3}{D_2}} \tag{A1}$$

$$\beta_2 = \sqrt{\sqrt{\beta_1^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\beta^4}{D_2}} - \beta_1^2\frac{D_3}{D_2}} \tag{A2}$$

Eqs. (A1) and (A2) can be rewritten as

$$D_1\beta_1^4 - 2D_3\beta_1^2\alpha_2^2 + D_2\alpha_2^4 - \beta^4 = 0 \tag{A3}$$

$$D_1\beta_2^4 + 2D_3\beta_2^2\beta_1^2 + D_2\beta_1^4 - \beta^4 = 0 \tag{A4}$$

Similarly, Eqs. (15a) and (15b) can also be rewritten as

$$D_1\alpha_1^4 + 2D_3\alpha_1^2\alpha_2^2 + D_2\alpha_2^4 - \beta^4 = 0 \tag{A5}$$

$$D_1\alpha_1^4 - 2D_3\alpha_1^2\beta_2^2 + D_2\beta_2^4 - \beta^4 = 0 \tag{A6}$$

Eqs. (17) and (18) can be changed to

$$\phi(x) = \phi_1(x) + \phi_2(x) \tag{A7}$$

$$\psi(y) = \psi_1(y) + \psi_2(y) \tag{A8}$$

where

$$\phi_1(x) = A_1\cos\alpha_1 x + B_1\sin\alpha_1 x, \quad \phi_2(x) = C_1\cosh\beta_1 x + H_1\sinh\beta_1 x \tag{A9}$$

$$\psi_1(y) = A_2\cos\alpha_2 y + B_2\sin\alpha_2 y, \quad \psi_2(y) = C_2\cosh\beta_2 y + H_2\sinh\beta_2 y \tag{A10}$$

By substituting Eqs. (A7) and (A8) into the left side of Eq. (7), one can obtain

$$\begin{aligned}
&D_1\phi^{(IV)}\psi + 2D_3\phi''\psi'' + D_2\phi\psi^{(IV)} - \beta^4\phi\psi \\
&= (D_1\alpha_1^4 + 2D_3\alpha_1^2\alpha_2^2 + D_2\alpha_2^4 - \beta^4)\phi_1\psi_1 \\
&\quad + (D_1\alpha_1^4 - 2D_3\alpha_1^2\beta_2^2 + D_2\beta_2^4 - \beta^4)\phi_1\psi_2 \\
&\quad + (D_1\beta_1^4 - 2D_3\beta_1^2\alpha_2^2 + D_2\alpha_2^4 - \beta^4)\phi_2\psi_1 \\
&\quad + (D_1\beta_2^4 + 2D_3\beta_2^2\beta_1^2 + D_2\beta_1^4 - \beta^4)\phi_2\psi_2
\end{aligned} \tag{A11}$$

Substituting Eqs. (A3)–(A6) into Eq. (A11) yields

$$D_1\phi^{(IV)}\psi + 2D_3\phi''\psi'' + D_2\phi\psi^{(IV)} - \beta^4\phi\psi = 0 \tag{A12}$$

Thus the expression (6) is the solution of Eq. (5).

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