Fuzzy Set Theory—and Its Applications, Fourth Edition

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H.-J. Zimmermann

Springer Science+Business Media, LLC

Library of Congress Cataloging-in-Publication Data

Zimmermann, H.-J. (Hans-Jürgen), 1934-
Fuzzy set theory--and its applications / H.-J. Zimmermann.--4th ed.
p. cm.
Includes bibliographical references and index.
ISBN 978-94-010-3870-6 ISBN 978-94-010-0646-0 (eBook)
DOI 10.1007/978-94-010-0646-0
1. Fuzzy sets. 2. Operations research. I. Title.

QA248 .Z55 2001
511.3'22--dc21

2001038123

Printed on acid-free paper.

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Foreword

As its name implies, the theory of fuzzy sets is, basically, a theory of graded concepts—a theory in which everything is a matter of degree or, to put it figuratively, everything has elasticity.

In the two decades since its inception, the theory has matured into a wide-ranging collection of concepts and techniques for dealing with complex phenomena that do not lend themselves to analysis by classical methods based on probability theory and bivalent logic. Nevertheless, a question that is frequently raised by the skeptics is: Are there, in fact, any significant problem-areas in which the use of the theory of fuzzy sets leads to results that could not be obtained by classical methods?

Professor Zimmermann's treatise provides an affirmative answer to this question. His comprehensive exposition of both the theory and its applications explains in clear terms the basic concepts that underlie the theory and how they relate to their classical counterparts. He shows through a wealth of examples the ways in which the theory can be applied to the solution of realistic problems, particularly in the realm of decision analysis, and motivates the theory by applications in which fuzzy sets play an essential role.

An important issue in the theory of fuzzy sets that does not have a counterpart in the theory of crisp sets relates to the combination of fuzzy sets through disjunction and conjunction or, equivalently, union and intersection. Professor Zimmermann and his associates at the Technical University of Aachen have made many important contributions to this problem and were the first to introduce the concept of a parametric family of connectives that can be chosen to fit a particular application. In recent years, this issue has given rise to an extensive literature dealing with *t*-norms and related concepts that link some aspects of the theory of fuzzy sets to the theory of probabilistic metric spaces developed by Karl Menger.

Another important issue addressed in Professor Zimmermann's treatise relates to the distinction between the concepts of probability and possibility, with the latter concept having a close connection with that of membership in a fuzzy set. The concept of possibility plays a particularly important role in the representation of meaning, in the management of uncertainty in expert systems, and in applications of the theory of fuzzy sets to decision analysis.

As one of the leading contributors to and practitioners of the use of fuzzy sets in decision analysis, Professor Zimmermann is uniquely qualified to address the complex issues arising in fuzzy optimization problems and, especially, fuzzy mathematical programming and multicriterion decision making in a fuzzy environment. His treatment of these topics is comprehensive, up-to-date, and illuminating.

In sum, Professor Zimmermann's treatise is a major contribution to the literature of fuzzy sets and decision analysis. It presents many original results and incisive analyses. And, most importantly, it succeeds in providing an excellent introduction to the theory of fuzzy sets—an introduction that makes it possible for an uninitiated reader to obtain a clear view of the theory and learn about its applications in a wide variety of fields.

The writing of this book was a difficult undertaking. Professor Zimmermann deserves to be congratulated on his outstanding accomplishment and thanked for contributing so much over the past decade to the advancement of the theory of fuzzy sets as a scientist, educator, administrator, and organizer.

L.A. Zadeh

Preface

Since its inception 20 years ago, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of this theory can be found, for example, in artificial intelligence, computer science, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Theoretical advances have been made in many directions. In fact it is extremely difficult for a newcomer to the field or for somebody who wants to apply fuzzy set theory to his problems to recognize properly the present "state of the art." Therefore, many applications use fuzzy set theory on a much more elementary level than appropriate and necessary. On the other hand, theoretical publications are already so specialized and assume such a background in fuzzy set theory that they are hard to understand. The more than 4,000 publications that exist in the field are widely scattered over many areas and in many journals. Existing books are edited volumes containing specialized contributions or monographs that focus only on specific areas of fuzzy sets, such as pattern recognition [Bezdek 1981], switching functions [Kandel and Lee 1979], or decision making [Kickert 1978]. Even the excellent survey book by Dubois and Prade [1980a] is primarily intended as a research compendium for insiders rather than an introduction to fuzzy set theory or a textbook. This lack of a comprehensive and modern text is particularly recognized by newcomers to the field and by those who want to teach fuzzy set theory and its applications.

The primary goal of this book is to help to close this gap—to provide a textbook for courses in fuzzy set theory and a book that can be used as an introduction.

One of the areas in which fuzzy sets have been applied most extensively is in modeling for managerial decision making. Therefore, this area has been selected for more detailed consideration. The information has been divided into two volumes. The first volume contains the basic theory of fuzzy sets and some areas of application. It is intended to provide extensive coverage of the theoretical and applicational approaches to fuzzy sets. Sophisticated formalisms have not been included. I have tried to present the basic theory and its extensions in enough detail to be comprehended by those who have not been exposed to fuzzy set theory. Examples and exercises serve to illustrate the concepts even more clearly. For the interested or more advanced reader, numerous references to recent literature are included that should facilitate studies of specific areas in more detail and on a more advanced level.

The second volume is dedicated to the application of fuzzy set theory to the area of human decision making. It is self-contained in the sense that all concepts used are properly introduced and defined. Obviously this cannot be done in the same breadth as in the first volume. Also the coverage of fuzzy concepts in the second volume is restricted to those that are directly used in the models of decision making.

It is advantageous but not absolutely necessary to go through the first volume before studying the second. The material in both volumes has served as texts in teaching classes in fuzzy set theory and decision making in the United States and in Germany. Each time the material was used, refinements were made, but the author welcomes suggestions for further improvements.

The target groups were students in business administration, management science, operations research, engineering, and computer science. Even though no specific mathematical background is necessary to understand the books, it is assumed that the students have some background in calculus, set theory, operations research, and decision theory.

I would like to acknowledge the help and encouragement of all the students, particularly those at the Naval Postgraduate School in Monterey and at the Institute of Technology in Aachen (F.R.G.), who improved the manuscripts before they became textbooks. I also thank Mr. Hintz, who helped to modify the different versions of the book, worked out the examples, and helped to make the text as understandable as possible. Ms. Grefen typed the manuscript several times without losing her patience. I am also indebted to Kluwer Academic Publishers for making the publication of this book possible.

H.-J. Zimmermann

Preface for the Revised Edition

Since this book was first published in 1985, Fuzzy Set Theory has had an unexpected growth. It was further developed theoretically and it was applied to new areas. A number of very good books have appeared, primarily dedicated to special areas such as Possibility Theory [Dubois and Prade 1988a], Fuzzy Control [Sugeno 1985a; Pedrycz 1989], Behavioral and Social Sciences [Smithson 1987], and others have been published. Many new edited volumes, either dedicated to special areas or with a much wider scope, have been added to the existing ones. Thousands of articles have been published on fuzzy sets in various journals. Successful real applications of fuzzy set theory have also increased in number and in quality. In particular, applications of fuzzy control, fuzzy computers, expert system shells with capabilities to process fuzzy information, and fuzzy decision support systems have become known and have partly already proven their superiority over more traditional tools.

One thing, however, does not seem to have changed since 1985: access to the area has not become easier for newcomers. I do not know of any introductory yet comprehensive book or textbook that will facilitate entering into the area of fuzzy sets or that can be used in classwork.

I am, therefore, very grateful to Kluwer Academic Publishers for having agreed to publish a revised edition of the book, which four times has already been printed without improvement. In this revised edition all typing and other errors have been eliminated. All chapters have been updated. The chapters on possibility theory (8), on fuzzy logic and approximate reasoning (9), on expert systems and fuzzy control (10), on decision making (12), and on fuzzy set models in operations research (13) have been restructured and rewritten. Exercises have been added to almost all chapters and a teacher's manual is available on request.

The intention of the book, however, has not changed: While the second volume

[Zimmermann 1987] focuses on decision making and expert systems and introduces fuzzy set theory only where and to the extent that it is needed, this book tries to offer a didactically prepared text which requires hardly any special mathematical background of the reader. It tries to introduce fuzzy set theory as comprehensively as possible, without delving into very theoretical areas or presenting any mathematical proofs which do not contribute to a better understanding. It rather offers numerical examples wherever possible. I would like to thank very much Mr. C. von Altrock, Ms. B. Lelke, Mr. R. Weber, and Dr. B. Werners for their active participation in preparing this revised edition. Mr. Andrée and Mr. Lehmann kindly prepared the figures. Ms. Oed typed and retyped manuscripts over and over again and helped us to arrive at the final manuscript of the book. We are all obliged to Kluwer Academic Publishers for the opportunity to publish this volume and for the good cooperation in preparing it.

H.-J. Zimmermann

Preface to the Third Edition

The development of fuzzy set theory to fuzzy technology during the first half of the 1990s has been very fast. More than 16,000 publications have appeared since 1965. Most of them have advanced the theory in many areas. Quite a number of these publications describe, however, applications of fuzzy set theory to existing methodology or to real problems. In addition, the transition from fuzzy set theory to fuzzy technology has been achieved by providing numerous software and hardware tools that considerably improve the design of fuzzy systems and make them more applicable in practice. Since 1994, fuzzy set theory, artificial neural nets, and genetic algorithms have also moved closer together and are now normally called "computational intelligence." All these changes have made this technology more powerful but also more complicated and have raised the "entrance barrier" even higher. This is particularly regrettable since more and more universities and other educational institutions are including fuzzy set theory in their programs. In some countries, a large number of introductory books have been published; in Germany, for instance, 25 such books were published in 1993 and 1994. English textbooks, however, are still very much lacking.

Therefore, I appreciate very much that Kluwer Academic Publishers has agreed to publish a third edition of this book, which updates the second revised edition.

New developments, to the extent that they are relevant for a basic textbook, have been included. All chapters have been updated. Chapters 9, 10, 11, and 12 have been completely rewritten. Nevertheless, I have tried not to let the book grow beyond a basic textbook. To reconcile the conflict between the nature of a textbook and the fast growth of the area, many references have been added to facilitate deeper insights for the interested reader.

I would like to thank Mr. Tore Grünert for his active participation and contributions, particularly to chapter 11, and all my coworkers for helping to proofread the book and to prepare new figures. We all hope that this third edition will benefit future students and accelerate the broader acceptance of fuzzy set theory.

Aachen, April 1995

H.-J. Zimmermann

Preface to the Fourth Edition

The new Millennium starts with over 30,000 publications in the area of "computational intelligence" or "soft computing". These are terms which have been coined in the first half of the 90s, when fuzzy set theory, neural networks and evolutionary computing joined forces because they felt that there were strong synergies between these areas. This is certainly true, in spite of the fact, that evolutionary computing has its strength in optimization, neural nets are particularly strong in pattern recognition and automatic learning, whereas fuzzy set theory has its strength in modeling, interfacing humans with computers and modeling certain uncertainties. Particularly between fuzzy set theory and neural nets the synergies have been used to develop hybrid models and methods, that combine the strengths of both of these areas.

Nevertheless, all three areas are continuing to develop new approaches in their own areas. For a textbook it would be inadequate to cover the basics of fuzzy set theory and also the vast area of computational intelligence or hybrid fuzzy-neuro methods or it would have to do this on a very superficial level. Hence, this book is restricted to the theory and application of fuzzy set theory only.

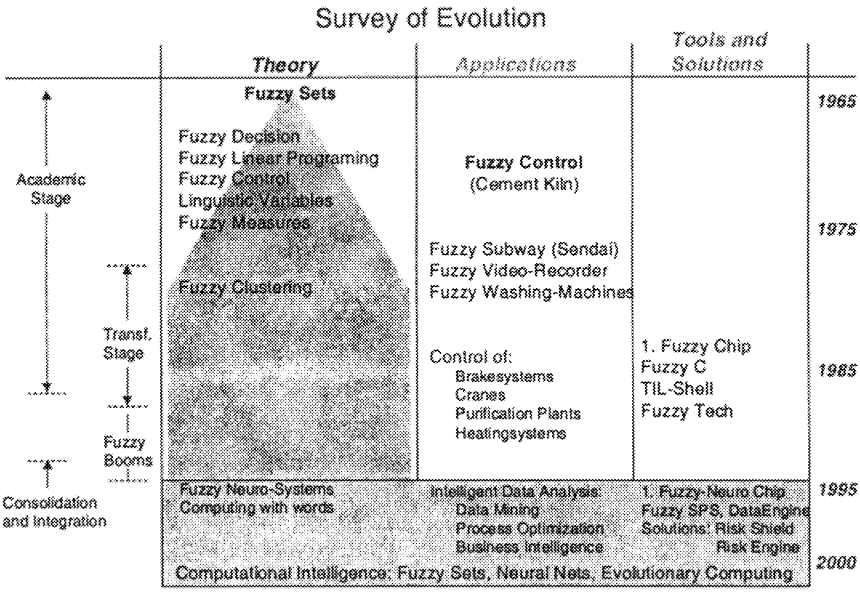
Apart from the convergence of above mentioned three areas two main developments can probably be observed for fuzzy sets:

1. There is a widening gap between the mathematics of fuzzy set theory and fuzzy technology, as the more applied version of fuzzy set theory. The still very strong activities in the theoretical direction lead to more and more very specific mathematical developments, which is natural, legitimate and certainly also important. The applicational relevance of these research results, however, is often not obvious and only perceivable by very advanced and

specialized theoreticians. New developments in fuzzy technology follow more the needs of a changed industrial environment.

2. One of the major changes in the industrial environment since the early 90s is, that we have moved from a situation of a lack of (electronically readable) data to one of an abundance of such data. Together with the dramatic increase in the power of electronic data processing and web-technology this has lead in fuzzy technology from a focus in modeling to a concentration in complexity reduction, i.e. pattern recognition, data mining and automatic knowledge discovery. This situation is mirrored in this edition of the book by an extension of the chapter on data mining and a new chapter on fuzzy sets in data bases.

The following figure indicates the development of fuzzy set theory from another point of view:



As shown there, the time lag between theory, application, and the development of a fuzzy technology (with efficient CASE-tools for the development of fuzzy systems), was, roughly speaking, ten years each.

This was valid until the first "fuzzy booms" occurred in the first half of the 90s. Until then the development of applications and technology centered very much around fuzzy control, a concept that was very applicable, easy to understand, and, therefore, attractive to many industrial practitioners and the broad public.

Since the start of computational intelligence theoretical as well as application-oriented developments have become much more diversified and clear lead-times between theoretical development and application can no longer be recognized.

I have used the opportunity of a fourth edition of this textbook, for which I am very grateful to Kluwer Academic Publishers, to adapt the book to the new developments, without exceeding the scope of a basic textbook, as follows:

All chapters have been up-dated. The scope of part I has only been extended with respect to t-norms, other operators and uncertainty modeling because I am convinced that chapters 2 to 7 are still sufficient as a mathematical basis to understand all new developments in this area and also for part II of the book, where the major changes and extensions of this edition can be found:

In chapter 10 the modeling of uncertainty in expert systems was extended because this component has gained importance in practice.

In chapter 11 primarily a section for defuzzification has been added for the same reason.

Chapter 12 has been added because the application of fuzzy technology in information processing is already important and will certainly increase in importance in the future.

Chapter 13 has been extended by explaining new methodological developments in dynamic fuzzy data analysis, which will also be of growing importance in the future.

Eventually applications in chapter 15 have been completely restructured by deleting some, adding others and classifying all of them differently. This was necessary because the focus of applications here changed, for reasons explained in this chapter, strongly from "engineering intelligence" to "business intelligence".

Of course, the index and the references have also been updated and extended.

This time I would like to thank again Kluwer Academic Publishers for giving me the chance of a fourth edition and Dr. Angstenberger for her excellent research cooperation and for letting me use one application from her book.

In particular, I would like to thank Ms. Katja Palczynski for her outstanding help to get the manuscripts ready for the publisher.

I hope that this new edition of my textbook will help to keep respective courses in universities and elsewhere up-to-date and challenging and motivating for students as well as professors. It may also be useful for practitioners that want to up-date their knowledge of fuzzy technology and look for new applications in their area.

Aachen, April 2001

H.-J. Zimmermann

1 INTRODUCTION TO FUZZY SETS

1.1 Crispness, Vagueness, Fuzziness, Uncertainty

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. By crisp we mean dichotomous, that is, yes-or-no-type rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false—and nothing in between. In set theory, an element can either belong to a set or not; and in optimization, a solution is either feasible or not. Precision assumes that the parameters of a model represent exactly either our perception of the phenomenon modeled or the features of the real system that has been modeled. Generally, precision also implies that the model is unequivocal, that is, that it contains no ambiguities.

Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known, and that there are no doubts about their values or their occurrence. If the model under consideration is a formal model [Zimmermann 1980, p. 127], that is, if it does not pretend to model reality adequately, then the model assumptions are in a sense arbitrary, that is, the model builder can freely decide which model characteristics he chooses. If, however, the model or theory asserts factuality [Popper 1959; Zimmermann 1980], that is, if conclusions drawn from these models have a bearing on reality and are supposed to model reality adequately, then the modeling language has to be suited to model the characteristics of the situation under study appropriately.

The utter importance of the modeling language is recognized by Apostel, when he says:

> The relationship between formal languages and domains in which they have models must in the empirical sciences necessarily be guided by two considerations that are by no means as important in the formal sciences:
> (a) The relationship between the language and the domain must be closer because they are in a sense produced through and for each other;
> (b) extensions of formalisms and models must necessarily be considered because everything introduced is introduced to make progress in the description of the objects studied. Therefore we should say that the formalization of the concept of approximate constructive necessary satisfaction is the main task of semantic study of models in the empirical sciences. [Apostel 1961, p. 26]

Because we request that a modeling language be unequivocal and nonredundant on one hand and, at the same time, catch semantically in its terms all that is important and relevant for the model, we seem to have the following problem. Human thinking and feeling, in which ideas, pictures, images, and value systems are formed, first of all certainly has more concepts or comprehensions than our daily language has words. If one considers, in addition, that for a number of notions we use several words (synonyms), then it becomes quite obvious that the power (in a set-theoretic sense) of our thinking and feeling is much higher than the power of a living language. If in turn we compare the power of a living language with the logical language, then we will find that logic is even poorer. Therefore it seems to be impossible to guarantee a one-to-one mapping of problems and systems in our imagination and in a model using a mathematical or logical language.

One might object that logical symbols can arbitrarily be filled with semantic contents and that by doing so the logical language becomes much richer. It will be shown that it is very often extremely difficult to appropriately assign semantic contents to logical symbols.

The usefulness of the mathematical language for modeling purposes is undisputed. However, there are limits to the usefulness and the possibility of using classical mathematical language, based on the dichotomous character of set theory, to model particular systems and phenomena in the social sciences: "There is no idea or proposition in the field, which can not be put into mathematical language, although the utility of doing so can very well be doubted" [Brand 1961]. Schwarz [1962] brings up another argument against the nonreflective use of mathematics when he states: "An argument, which is only convincing if it is precise loses all its force if the assumptions on which it is based are slightly changed, while an argument, which is convincing but imprecise may well be stable under small perturbations of its underlying axioms." For factual models or modeling languages, two major complications arise:

1. Real situations are very often not crisp and deterministic, and they cannot be described precisely.
2. The complete description of a real system often would require far more detailed data than a human being could ever recognize simultaneously, process, and understand.

This situation has already been recognized by thinkers in the past. In 1923 the philosopher B. Russell [1923] referred to the first point when he wrote:

> All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

L. Zadeh referred to the second point when he wrote, "As the complexity of a system increases, our ability to make precise and yet significant statements about its behaviour diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics." [Zadeh 1973a]

Let us consider characteristic features of real-world systems again: Real situations are very often uncertain or vague in a number of ways. Due to lack of information, the future state of the system might not be known completely. This type of uncertainty (stochastic character) has long been handled appropriately by probability theory and statistics. This Kolmogoroff-type probability is essentially frequentistic and is based on set-theoretic considerations. Koopman's probability refers to the truth of statements and therefore is based on logic. In both types of probabilistic approaches, however, it is assumed that the events (elements of sets) or the statements, respectively, are well defined. We shall call this type of uncertainty or vagueness *stochastic uncertainty* in contrast to the vagueness concerning the description of the semantic meaning of the events, phenomena, or statements themselves, which we shall call *fuzziness*.

Fuzziness can be found in many areas of daily life, such as in engineering [see, for instance, Blockley 1980], medicine [see Vila and Delgado 1983], meteorology [Cao and Chen 1983], manufacturing [Mamdani 1981], and others. It is particularly frequent, however, in all areas in which human judgment, evaluation, and decisions are important. These are the areas of decision making, reasoning, learning, and so on. Some reasons for this fuzziness have already been mentioned. Others are that most of our daily communication uses "natural languages," and a good part of our thinking is done in it. In these natural languages, the meaning of words is very often vague. The meaning of a word might even be well defined, but when using the word as a label for a set, the boundaries within which objects do or do not belong to the set become fuzzy or vague. Examples are words such as "birds" (how about penguins, bats, etc.?) or "red roses," but also terms such as "tall men," "beautiful women," and "creditworthy customers." In this context we can probably distinguish two kinds of fuzziness with respect to their origins: intrinsic fuzziness and informational fuzziness. The former is the fuzziness to which Russell's remark referred, and it is illustrated by "tall men." This term is fuzzy because the meaning of tall is fuzzy and dependent on the context (height of observer, culture, etc.). An example of the latter is the term "creditworthy customers": A creditworthy customer can possibly be described completely and crisply if we use a large number of descriptors. These descriptors are more, however, than a human being could handle simultaneously. Therefore the term, which in psychology is called a "subjective category," becomes fuzzy. One could imagine that the subjective category "creditworthiness" is decomposed into two smaller subjective categories, each of which needs fewer descriptors to be completely described. This process of decomposition could be continued until the descriptions of the subjective categories generated are reasonably defined. On the other hand, the notion "creditworthiness" could be constructed by starting with the smallest subjective subcategories and aggregating them hierarchically.

For creditworthiness the concept structure shown in figure 1–1, which has a symmetrical structure, was developed in consultation with 50 credit clerks of banks.

Credit experts distinguish between the financial basis and the personality of an applicant. "Financial basis" comprises all realities, movables, assets, liquid funds, and others. The evaluation of the economic situation depends on the actual securities, that is, the difference between property and debts, and on the liquidity, that is, the continuous difference between income and expenses.

On the other hand, "personality" denotes the collection of traits by which a potent and serious person is distinguished. The achievement potential is based on mental and physical capacity as well as on the individual's motivation. The business conduct includes economical standards. While the former means the setting of realistic goals, reasonable planning, and criteria of economic success, the latter is directed toward the applicant's disposition to obey business laws and mutual agreements. Hence a credit-worthy person lives in secure circumstances and guarantees a successful, profit-oriented cooperation (see figure 1–1).

Before turning to fuzzy set theory it should, however, be stressed that uncertainty is a multi-facetted phenomenon and that the modeling of it in application-oriented models requires considerable investigations before we start the modeling

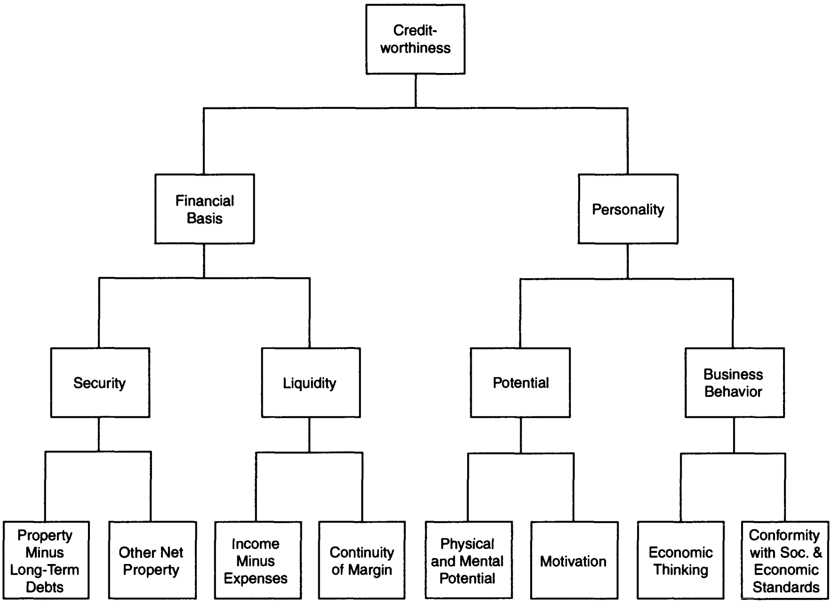


Figure 1–1. Concept hierarchy of creditworthiness.

process. Also the available modeling tools do not only include probability theory and fuzzy set theory. We shall consider this fact in more detail in chapter 8.

In chapter 16 we will return to this figure and elaborate on the type of aggregation.

1.2 Fuzzy Set Theory

The first publications in fuzzy set theory by Zadeh [1965] and Goguen [1967, 1969] show the intention of the authors to generalize the classical notion of a set and a proposition [statement] to accommodate fuzziness in the sense described in section 1.1.

Zadeh [1965, p. 339] writes, "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual frame-work which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables."

"Imprecision" here is meant in the sense of vagueness rather than the lack of knowledge about the value of a parameter (as in tolerance analysis). Fuzzy set theory provides a strict mathematical framework (there is nothing fuzzy about fuzzy set theory!) in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language well suited for situations in which fuzzy relations, criteria, and phenomena exist.

Fuzziness has so far not been defined uniquely semantically, and probably never will be. It will mean different things, depending on the application area and the way it is measured. In the meantime, numerous authors have contributed to this theory. In 1984, as many as 4,000 publications have already existed and in 2000 there were already more than 30,000.

The specialization of those publications conceivably increases, making it more and more difficult for newcomers to this area to find a good entry and to understand and appreciate the philosophy, formalism, and applications potential of this theory. Roughly speaking, fuzzy set theory in the last two decades has developed along two lines:

1. As a formal theory that, when maturing, became more sophisticated and specified and was enlarged by original ideas and concepts as well as by "embracing" classical mathematical areas such as algebra, graph theory, topology, and so on by generalizing (fuzzifying) them.
2. As an application-oriented "fuzzy technology", i.e. as a tool for modeling, problem solving and data mining that has proven superior to existing methods in many cases and as an attractive "add-on" to classical approaches in other cases.

In this context it may be useful to cite and comment the *major goals* of this technology briefly and to correct the still very common view that fuzzy set theory or fuzzy technology is exclusively or primarily useful to model uncertainty:

a) Modeling of uncertainty

This is certainly the best known and oldest goal. I am not sure, however, whether it can (still) be considered to be the most important goal of fuzzy set theory. Uncertainty has been a very important topic for several centuries. There are numerous methods and theories which claim to be the only proper tool to model uncertainties. In general, however, they do not even define sufficiently or only in a very specific and limited sense what is meant by "uncertainty". I believe that uncertainty, if considered as a subjective phenomenon, can and ought to be modeled by very different theories, depending on the causes of uncertainty, the type and quantity of available information, the requirements of the observer etc. In this sense fuzzy set theory is certainly also one of the theories which can be used to model specific types of uncertainty under specific types of circumstances. It might then compete with other theories, but it might also be the most appropriate way to model this phenomenon for well-specified situations. It would certainly exceed the scope of this article to discuss this question in detail here [Zimmermann 1997].

b) Relaxation

Classical models and methods are normally based on dual logic. They, therefore, distinguish between feasible and infeasible, belonging to a cluster or not, optimal or suboptimal etc. Often this view does not capture reality adequately. Fuzzy set theory has been used extensively to relax or generalize classical methods from a dichotomous to a gradual character. Examples of this are fuzzy mathematical programming [Zimmermann 1996], fuzzy clustering [Bezdek and Pal 1992], fuzzy Petri Nets [Lipp et al. 1989], fuzzy multi criteria analysis [Zimmermann 1986].

c) Compactification

Due to the limited capacity of the human short term memory or of technical systems it is often not possible to either store all relevant data, or to present masses of data to a human observer in such a way, that he or she can perceive the information contained in these data. Fuzzy technology has been used to reduce the complexity of data to an acceptably degree usually either via linguistic variables or via fuzzy data analysis (fuzzy clustering etc.).

d) Meaning Preserving Reasoning

Expert system technology has already been used since two decades and has led in many cases to disappointment. One of the reasons for this might be, that expert systems in their inference engines, when they are based on dual logic, perform symbol processing (truth values true or false) rather than knowledge processing. In approximate reasoning meanings are attached to words and sentences via linguistic variables. Inference engines then have to be able to process meaningful linguistic expressions, rather than symbols, and arrive at membership functions of fuzzy sets, which can then be retranslated into words and sentences via linguistic approximation.

e) Efficient Determination of Approximate Solutions

Already in the 70s Prof. Zadeh expressed his intention to have fuzzy set theory considered as a tool to determine approximate solutions of real problems in an efficient or affordable way. This goal has never really been achieved successfully. In the recent past, however, cases have become known which are very good examples for this goal. Bardossy [1996], for instance, showed in the context of water flow modeling that it can be much more efficient to use fuzzy rule based systems to solve the problems than systems of differential equations. Comparing the results achieved by these two alternative approaches showed that the accuracy of the results was almost the same for all practical purposes. This is particularly true if one considers the inaccuracies and uncertainties contained in the input data.

It seems desirable that an introductory textbook be available to help students get started and find their way around. Obviously, such a textbook cannot cover the entire body of the theory in appropriate detail. The present book will therefore proceed as follows:

Part I of this book, containing chapters 2 to 8, will develop the formal framework of fuzzy mathematics. Due to space limitations and for didactical reasons, two restrictions will be observed:

1. Topics that are of high mathematical interest but require a very solid mathematical background and those that are not of obvious relevance to applications will not be discussed.
2. Most of the discussion will proceed along the lines of the early concepts of fuzzy set theory. At appropriate times, however, the additional potential of fuzzy set theory that arises by using other axiomatic frameworks resulting in other operators will be indicated or described. The character of these chapters will obviously have to be formal.

Part II of the book, chapters 9 to 16, will then survey the most interesting applications of fuzzy set theory. At that stage the student should be in a position to recognize possible extensions and improvements of the applications presented.

I FUZZY MATHEMATICS

This first part of this book is devoted to the formal framework of the theory of fuzzy sets. Chapter 2 provides basic definitions of fuzzy sets and algebraic operations that will then serve for further considerations. Even though we shall use one version of terminology and one set of symbols consistently throughout the book, alternative ways of denoting fuzzy sets will be mentioned because they have become common. Chapter 3 extends the basic theory of fuzzy sets by introducing additional concepts and alternative operators. Chapter 4 is devoted to fuzzy measures, measures of fuzziness, and other important measures that are needed for applications presented either in Part II of this book or in the second volume on decision making in a fuzzy environment. Chapter 5 introduces the extension principle, which will be very useful for the following chapters and covers fuzzy arithmetic. Chapters 6 and 7 will then treat fuzzy relations, graphs, and functions. Chapter 8 focuses on uncertainty modeling and some special topics, such as the relationship between fuzzy set theory, probability theory, and other classical areas.

2 FUZZY SETS—BASIC DEFINITIONS

2.1 Basic Definitions

A *classical* (crisp) *set* is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable, or overcountable. Each single element can either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belongs to A" is true, whereas in the latter case this statement is false.

Such a classical set can be described in different ways: one can either enumerate (list) the elements that belong to the set; describe the set analytically, for instance, by stating conditions for membership ($A = \{x|x \leq 5\}$); or define the member elements by using the characteristic function, in which 1 indicates membership and 0 nonmembership. For a fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set.

Definition 2–1

If X is a collection of objects denoted generically by x, then a *fuzzy set* $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ that maps X to the membership space M (When M contains only the two points 0 and 1, $\tilde{A}$ is nonfuzzy and $\mu_{\tilde{A}}(x)$ is identical to the characteristic function of a nonfuzzy set). The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed.

Example 2–1a

A realtor wants to classify the house he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in it. Let $X = \{1, 2, 3, 4, \ldots, 10\}$ be the set of available types of houses described by x = number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a four-person family" may be described as

$$\tilde{A} = \{(1, .2), (2, .5), (3, .8), (4, 1), (5, .7), (6, .3)\}$$

In the literature one finds different ways of denoting fuzzy sets:

1. A fuzzy set is denoted by an ordered set of pairs, the first element of which denotes the element and the second the degree of membership (as in definition 2–1).

Example 2–1b

$\tilde{A}$ = "real numbers considerably larger than 10"

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

where

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le 10 \\ (1 + (x-10)^{-2})^{-1}, & x > 10 \end{cases}$$

Example 2–1c

$\tilde{A}$ = "real numbers close to 10"

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid \mu_{\tilde{A}}(x) = (1 + (x - 10^2)^{-1}\}$$

See figure 2–1.

2. A fuzzy set is represented solely by stating its membership function [for instance, Negoita and Ralescu 1975].

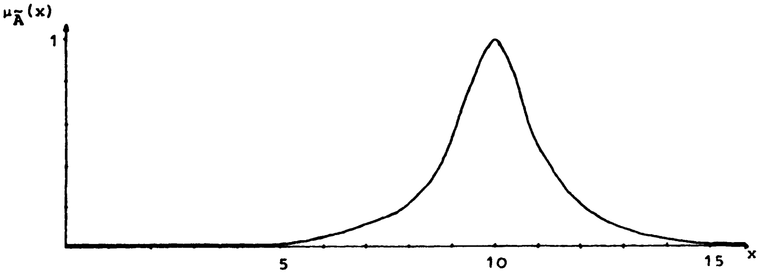


Figure 2–1. Real numbers close to 10.

3. $$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 \ldots = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i$$
$$\text{or } \int_X \mu_{\tilde{A}}(\underline{x})/x$$

Example 2–1d

$\tilde{A}$ = "integers close to 10"

$$\tilde{A} = 0.1/7 + 0.5/8 + 0.8/9 + 1/10 + 0.8/11 + 0.5/12 + 0.1/13$$

Example 2–1e

$\tilde{A}$ = "real numbers close to 10"

$$\tilde{A} = \int_{\mathbb{R}} \frac{1}{1+(x-10)^2} \bigg/ x$$

It has already been mentioned that the membership function is not limited to values between 0 and 1. If $\sup_x \mu_{\tilde{A}}(x) = 1$, the fuzzy set $\tilde{A}$ is called normal. A nonempty fuzzy set $\tilde{A}$ can always be normalized by dividing $\mu_{\tilde{A}}(x)$ by $\sup_x \mu_{\tilde{A}}(x)$: As a matter of convenience, we will generally assume that fuzzy sets are normalized. For the representation of fuzzy sets, we will use the notation 1 illustrated in examples 2–1b and 2–1c, respectively.

A fuzzy set is obviously a generalization of a classical set and the membership function a generalization of the characteristic function. Since we are generally referring to a universal (crisp) set X, some elements of a fuzzy set may have the degree of membership zero. Often it is appropriate to consider those elements of the universe that have a nonzero degree of membership in a fuzzy set.

Definition 2–2

The *support* of a fuzzy set $\tilde{A}$, $S(\tilde{A})$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Example 2–2

Let us consider example 2–1a again: The support of $S(\tilde{A}) = \{1, 2, 3, 4, 5, 6\}$. The elements (types of houses) $\{7, 8, 9, 10\}$ are not part of the support of $\tilde{A}$!

A more general and even more useful notion is that of an α-level set.

Definition 2–3

The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the *α-level set*:

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$$

$A'_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) > \alpha\}$ is called "strong α-level set" or "strong α-cut."

Example 2–3

We refer again to example 2–1a and list possible α-level sets:

$$\begin{aligned} A_{.2} &= \{1, 2, 3, 4, 5, 6\} \\ A_{.5} &= \{2, 3, 4, 5\} \\ A_{.8} &= \{3, 4\} \\ A_{1} &= \{4\} \end{aligned}$$

The strong α-level set for $\alpha = .8$ is $A'_{.8} = \{4\}$.

Convexity also plays a role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of the fuzzy set.

Definition 2–4

A fuzzy set $\tilde{A}$ is *convex* if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, x_1, x_2 \in X, \lambda \in [0,1]$$

Alternatively, a fuzzy set is convex if all α-level sets are convex.

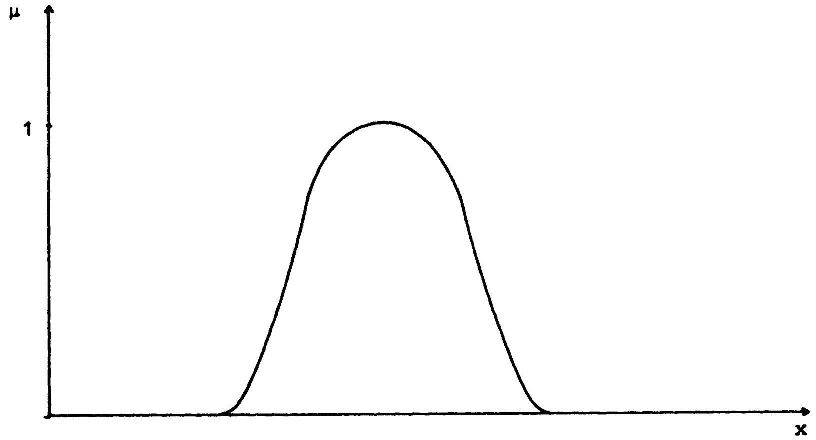


Figure 2–2a. Convex fuzzy set.

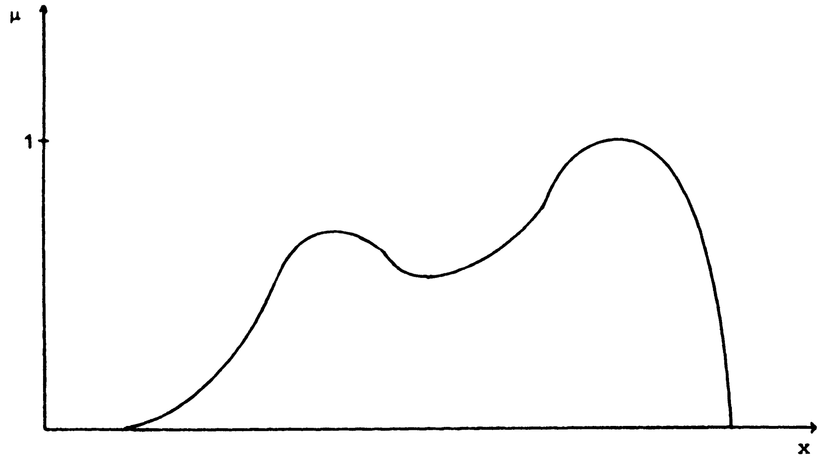


Figure 2–2b. Nonconvex fuzzy set.

Example 2–4

Figure 2–2a depicts a convex fuzzy set, whereas figure 2–2b illustrates a nonconvex fuzzy set.

One final feature of a fuzzy set, which we will use frequently in later chapters, is its cardinality or "power" [Zadeh 1981c].

Definition 2–5

For a finite fuzzy set $\tilde{A}$, the *cardinality* $|\tilde{A}|$ is defined as

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$$

$\|\tilde{A}\| = \dfrac{|\tilde{A}|}{|x|}$ is called the *relative cardinality* of $\tilde{A}$.

Obviously, the relative cardinality of a fuzzy set depends on the cardinality of the universe. So you have to choose the same universe if you want to compare fuzzy sets by their relative cardinality.

Example 2–5

For the fuzzy set "comfortable type of house for a four-person family" from example 2–1a, the cardinality is

$$|\tilde{A}| = .2 + .5 + .8 + 1 + .7 + .3 = 3.5$$

Its relative cardinality is

$$\|\tilde{A}\| = \frac{3.5}{10} = 0.35$$

The relative cardinality can be interpreted as the fraction of elements of X being in $\tilde{A}$, weighted by their degrees of membership in $\tilde{A}$. For infinite X, the cardinality is defined by $|\tilde{A}| = \int_x \mu_{\tilde{A}}(x)\, dx$. Of course, $|\tilde{A}|$ does not always exist.

2.2 Basic Set-Theoretic Operations for Fuzzy Sets

The membership function is obviously the crucial component of a fuzzy set. It is therefore not surprising that operations with fuzzy sets are defined via their membership functions. We shall first present the concepts suggested by Zadeh in 1965 [Zadeh 1965, p. 310]. They constitute a consistent framework for the theory of fuzzy sets. They are, however, not the only possible way to extend classical set theory consistently. Zadeh and other authors have suggested alternative or additional definitions for set-theoretic operations, which will be discussed in chapter 3.

Definition 2–6

The membership function $\mu_{\tilde{C}}(x)$ of the *intersection* $\tilde{C} = \tilde{A} \cap \tilde{B}$ is pointwise defined by

$$\mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad x \in X$$

Definition 2–7

The membership function $\mu_{\tilde{D}}(x)$ of the *union* $\tilde{D} = \tilde{A} \cup \tilde{B}$ is pointwise defined by

$$\mu_{\tilde{D}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad x \in X$$

Definition 2–8

The membership function of the *complement* of a normalized fuzzy set $\tilde{A}$, $\mu_{\complement\tilde{A}}(x)$ is defined by

$$\mu_{\complement A}(x) = 1 - \mu_{\tilde{A}}(x) \quad x \in X$$

Example 2–6

Let $\tilde{A}$ be the fuzzy set "comfortable type of house for a four-person family" from example 2–1a and $\tilde{B}$ be the fuzzy set "large type of house" defined as

$$\tilde{B} = \{(3, .2), (4, .4), (5, .6), (6, .8), (7, 1), (8, 1)\}$$

The intersection $\tilde{C} = \tilde{A} \cap \tilde{B}$ is then

$$\tilde{C} = \{(3, .2), (4, .4), (5, .6), (6, .3)\}$$

The union $\tilde{D} = \tilde{A} \cup \tilde{B}$ is

$$\tilde{D} = \{(1, .2), (2, .5), (3, .8), (4, 1), (5, .7), (6, .8), (7, 1), (8, 1)\}$$

The complement $\complement\tilde{B}$, which might be interpreted as "not large type of house," is

$$\complement\tilde{B} = \{(1, 1), (2, 1), (3, .8), (4, .6), (5, .4), (6, .2), (9, 1), (10, 1)\}$$

Example 2–7

Let us assume that

$\tilde{A}$ = "x is considerable larger than 10," and
$\tilde{B}$ = "x is approximately 11," characterized by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le 10 \\ (1 + (x - 10^{-2})^{-1} & x > 10 \end{cases}$$

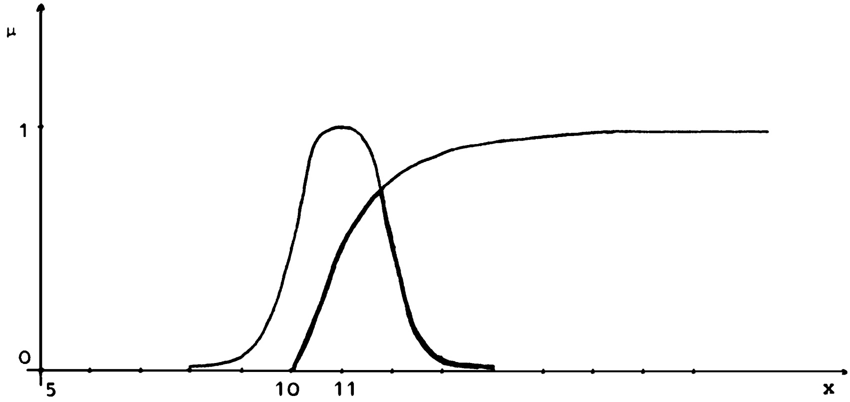


Figure 2–3. Union and intersection of fuzzy sets.

and

$$\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{B}}(x) = (1 + (x - 11)^4)^{-1}$$

Then

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \begin{cases} \min[(1+(x-10)^{-2})^{-1}, (1+(x-11)^4)^{-1}] & \text{for} \quad x > 10 \\ 0 & \text{for} \quad x \le 10 \end{cases}$$

(x is considerably larger than 10 and approximately 11)

$$\mu_{A \cup \tilde{B}}(x) = \max[(1+(x-10)^{-2})^{-1}, (1+(x-11)^4)^{-1}], \quad x \in X$$

Figure 2–3 depicts the above.

It has already been mentioned that min and max are not the only operators that could have been chosen to model the intersection or union, respectively, of fuzzy sets. The question arises, why those and not others? Bellman and Giertz addressed this question axiomatically in 1973 [Bellman and Giertz 1973, p. 151]. They argued from a logical point of view, interpreting the intersection as "logical and," the union as "logical or," and the fuzzy set $\tilde{A}$ as the statement "The element x belongs to set $\tilde{A}$," which can be accepted as more or less true. It is very instructive to follow their line of argument, which is an excellent example for an axiomatic justification of specific mathematical models. We shall therefore sketch their reasoning: Consider two statements, S and T, for which the truth values are

μ_S and μ_T, respectively, μ_S, $\mu_T \in [0, 1]$. The truth value of the "and" and "or" combination of these statements, $\mu(S \text{ and } T)$ and $\mu(S \text{ or } T)$, both from the interval [0, 1], are interpreted as the values of the membership functions of the intersection and union, respectively, of S and T. We are now looking for two real-valued functions f and g such that

$$\mu_{S \text{ and } T} = f(\mu_S, \mu_T)$$

$$\mu_{S \text{ or } T} = g(\mu_S, \mu_T)$$

Bellman and Giertz feel that the following restrictions are reasonably imposed on f and g:

i. f and g are nondecreasing and continuous in μ_S and μ_T.
ii. f and g are symmetric, that is,

$$f(\mu_S, \mu_T) = f(\mu_T, \mu_S)$$

$$g(\mu_S, \mu_T) = g(\mu_T, \mu_S)$$

iii. $f(\mu_S, \mu_S)$ and $g(\mu_S, \mu_S)$ are strictly increasing in μ_S.
iv. $f(\mu_S, \mu_T) \leq \min(\mu_S, \mu_T)$ and $g(\mu_S, \mu_T) \geq \max(\mu_S, \mu_T)$. This implies that accepting the truth of the statement "S and T" requires more, and accepting the truth of the statement "S or T" less than accepting S or T alone as true.
v. $f(1, 1) = 1$ and $g(0, 0) = 0$.
vi. Logically equivalent statements must have equal truth values, and fuzzy sets with the same contents must have the same membership functions, that is,

$$S_1 \text{ and } (S_2 \text{ or } S_3)$$

is equivalent to

$$(S_1 \text{ and } S_2) \text{ or } (S_1 \text{ and } S_3)$$

and therefore must be equally true.

Bellman and Giertz now formalize the above assumptions as follows: Using the symbols $\wedge$ for "and" (= intersection) and $\vee$ for "or" (= union), these assumptions amount to the following seven restrictions, to be imposed on the two commutative (see (ii)) and associative (see (vi)) binary compositions $\wedge$ and $\vee$ on the closed interval [0, 1], which are mutually distributive (see (vi)) with respect to one another.

1. $\mu_S \wedge \mu_T = \mu_T \wedge \mu_S$
   $\mu_S \vee \mu_T = \mu_T \vee \mu_S$
2. $(\mu_S \wedge \mu_T) \wedge \mu_U = \mu_S \wedge (\mu_T \wedge \mu_U)$
   $(\mu_S \vee \mu_T) \vee \mu_U = \mu_S \vee (\mu_T \vee \mu_U)$

3. $\mu_S \wedge (\mu_T \vee \mu_U) = (\mu_S \wedge \mu_T) \vee (\mu_S \wedge \mu_U)$
   $\mu_S \vee (\mu_T \wedge \mu_U) = (\mu_S \vee \mu_T) \wedge (\mu_S \vee \mu_U)$
4. $\mu_S \wedge \mu_T$ and $\mu_S \vee \mu_T$ are continuous and nondecreasing in each component
5. $\mu_S \wedge \mu_S$ and $\mu_S \vee \mu_S$ are strictly increasing in μ_S (see (iii))
6. $\mu_S \wedge \mu_T \leq \min(\mu_S, \mu_T)$
   $\mu_S \vee \mu_T \geq \max(\mu_S, \mu_T)$ (see (iv))
7. $1 \wedge 1 = 1$
   $0 \vee 0 = 0$ (see (v))

Bellman and Giertz then prove mathematically [see Bellman and Giertz 1973, p. 154] that

$$\mu_{S \wedge T} = \min(\mu_S, \mu_T) \quad \text{and} \quad \mu_{S \vee T} = \max(\mu_S, \mu_T)$$

For the complement, it would be reasonable to assume that if statement "*S*" is true, its complement "non *S*" is false, or if $\mu_S = 1$ then $\mu_{\text{non}S} = 0$ and vice versa. The function *h* (as complement in analogy of *f* and *g* for intersection and union) should also be continuous and monotonically decreasing, and we would like the complement of the complement to be the original statement (in order to be in line with traditional logic and set theory). These requirements, however, are not enough to determine uniquely the mathematical form of the complement. Bellman and Giertz require in addition that $\mu_{\bar{S}}(1/2) = 1/2$. Other assumptions are certainly possible and plausible.

Exercises

1. Model the following expressions as fuzzy sets:
   a. Large integers
   b. Very small numbers
   c. Medium-sized men
   d. Numbers approximately between 10 and 20
   e. High speeds for racing cars
2. Determine all α-level sets and all strong α-level sets for the following fuzzy sets:
   a. $\tilde{A} = \{(3, 1), (4, .2), (5, .3), (6, .4), (7, .6), (8, .8), (10, 1), (12, .8), (14, .6)\}$
   b. $\tilde{B} = \{(x, \mu_{\tilde{B}}(x) = (1 + (x - 10)^2)^{-1})\}$
      for $\alpha = .3, .5, .8$
   c. $\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) | x \in R\}$
      where $\mu_{\tilde{C}}(x) = 0$ for $x \leq 10$
      $\mu_{\tilde{C}}(x) = (1 + (x - 10)^{-2})^{-1}$ for $x > 10$

3. Which of the fuzzy sets of exercise 2 are convex and which are not?
4. Let $X = \{1, 2, \ldots, 10\}$. Determine the cardinalities and relative cardinalities of the following fuzzy sets:
   a. $\tilde{A}$ from exercise 2a
   b. $\tilde{B} = \{(2, .4), (3, .6), (4, .8), (5, 1), (6, .8), (7, .6), (8, .4)\}$
   c. $\tilde{C} = \{(2, .4), (4, .8), (5, 1), (7, .6)\}$
5. Determine the intersections and unions of the following fuzzy sets:
   a. The fuzzy sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ from exercise 4
   b. $\tilde{B}$ and $\tilde{C}$ from exercise 2
6. Determine the intersection and the union of the complements of fuzzy sets $\tilde{B}$ and $\tilde{C}$ from exercise 4.

3 EXTENSIONS

3.1 Types of Fuzzy Sets

In chapter 2, the basic definition of a fuzzy set was given and the original set-theoretic operations were discussed. The membership space was assumed to be the space of real numbers, membership functions were crisp functions, and the operations corresponded essentially to the operations of dual logic or Boolean algebra.

Different extensions of the basic concept discussed in chapter 2 are possible. They may concern the definition of a fuzzy set or they may concern the operations with fuzzy sets. With respect to the definition of a fuzzy set, different structures may be imposed on the membership space and different assumptions may be made concerning the membership function. These extensions will be treated in section 3.1.

It was assumed in chapter 2 that the logical "and" corresponds to the set-theoretic intersection, which in turn is modeled by the min-operator. The same type of relationship was assumed for the logical "or," the union, and the max-operator. Departing from the well-established systems of dual logic and Boolean algebra, alternative and additional definitions for terms such as intersection and union, for their interpretation as "and" and "or," and for their mathematical models can be conceived. These concepts will be discussed in section 3.2.

So far we have considered fuzzy sets with crisply defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [1973a, p. 52] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. If we call fuzzy sets, such as those considered so far, type 1 fuzzy sets, then a type 2 fuzzy set can be defined as follows.

Definition 3–1

A *type 2 fuzzy set* is a fuzzy set whose membership values are type 1 fuzzy sets on [0, 1].

The operations intersection, union, and complement defined so far are no longer adequate for type 2 fuzzy sets. We will, however, postpone the discussions for adequate operators until section 5.2, that is, until we have presented the extension principle, which shall prove very useful for this purpose. By the same token by which we introduced type 2 fuzzy sets, it could be argued that there is no obvious reason why the membership functions of type 2 fuzzy sets should be crisp. A natural extension of these type 2 fuzzy sets is therefore the definition of type m fuzzy sets.

Definition 3–2

A *type m fuzzy set* is a fuzzy set in X whose membership values are type $m - 1$, $m > 1$ fuzzy sets on [0, 1].

From a practical point of view, such type m fuzzy sets for large m (even for $m \geq 3$) are hard to deal with, and it will be extremely difficult or even impossible to measure them or to visualize them. We will, therefore, not even try to define the usual operations on them.

There have been other attempts to include vagueness that goes beyond the fuzziness of ordinary type 1 fuzzy sets. One example is the "stochastic fuzzy model" of Norwich and Turksen [1981, 1984]. Those authors were mainly concerned with the measurement and the scale level of membership functions. They view a fuzzy set as a family of random variables whose density functions are estimated by that stochasticity [Norwich and Turksen 1984, p. 21].

Hirota [1981] also considers fuzzy sets for which the "value of membership functions is a random variable."

Definition 3–3 [Hirota 1981, p. 35]

A *probabilistic set* A on X is defined by a defining function μ_A,

$$\mu_A : X \times \Omega \ni (x, \omega) \to \mu_A(x, \omega) \in \Omega_C$$

where $\mu_A(x, \cdot)$ is the (B, B_C)-measurable function for each fixed $x \in X$.

For Hirota, a probabilistic set A with the defining function $\mu_A(x, \omega)$ is contained in a probabilistic set B with $\mu_B(x, \omega)$ if for each $x \in X$ there exists an $E \in B$ that satisfies $P(E) = 1$ and $\mu_A(x, \omega) \le \mu_B(x, \omega)$ for all $\omega \in E$. (Ω, B, P) is called the parameter space.

One of the main advantages of the notion of probabilistic sets in modeling fuzzy and stochastic features of a system is asserted to be the applicability of moment analysis, that is, the possibility of computing moments such as expectation and variance. Figure 3–1 indicates the difference between the appearance of fuzzy sets and probabilistic sets [Hirota 1981, p. 33]. Of course, the mathematical properties of probabilistic sets differ from those of fuzzy sets, and so do the mathematical models for intersection, union, and so on.

A more general definition of a fuzzy set than is given in definition 2–1 is that of an L-fuzzy set [Goguen 1967; De Luca and Termini 1972]. In contrast to the above definition, the membership function of an L-fuzzy set maps into a partially ordered set, L. Since the interval [0, 1] is a poset (partially ordered set), the fuzzy set in definition 2–1 is a special L-fuzzy set.

Further attempts at representing vague and uncertain data with different types of fuzzy sets were made by Atanassov and Stoeva [Atanassov and Stoeva 1983; Atanassov 1986], who defined a generalization of the notion of fuzzy sets—the *intuitonistic* fuzzy sets—and by Pawlak [Pawlak 1982], who developed the theory of rough sets, where grades of membership are expressed by a concept of approximation.

Definition 3–4 [Atanassov and Stoeva 1983]

Given an underlying set X of objects, an *intuitonistic fuzzy set* (IFS) A is a set of ordered triples,

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$$

where $\mu_A(x)$ and $\nu_A(x)$ are functions mapping from X into [0, 1]. For each $x \in X$, $\mu_A(x)$ represents the degree of membership of the element x to the subset A of X, and $\nu_A(x)$ gives the degree of nonmembership. For the functions $\mu_A(x)$ and $\nu_A(x)$ mapping into [0, 1], the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$ holds.

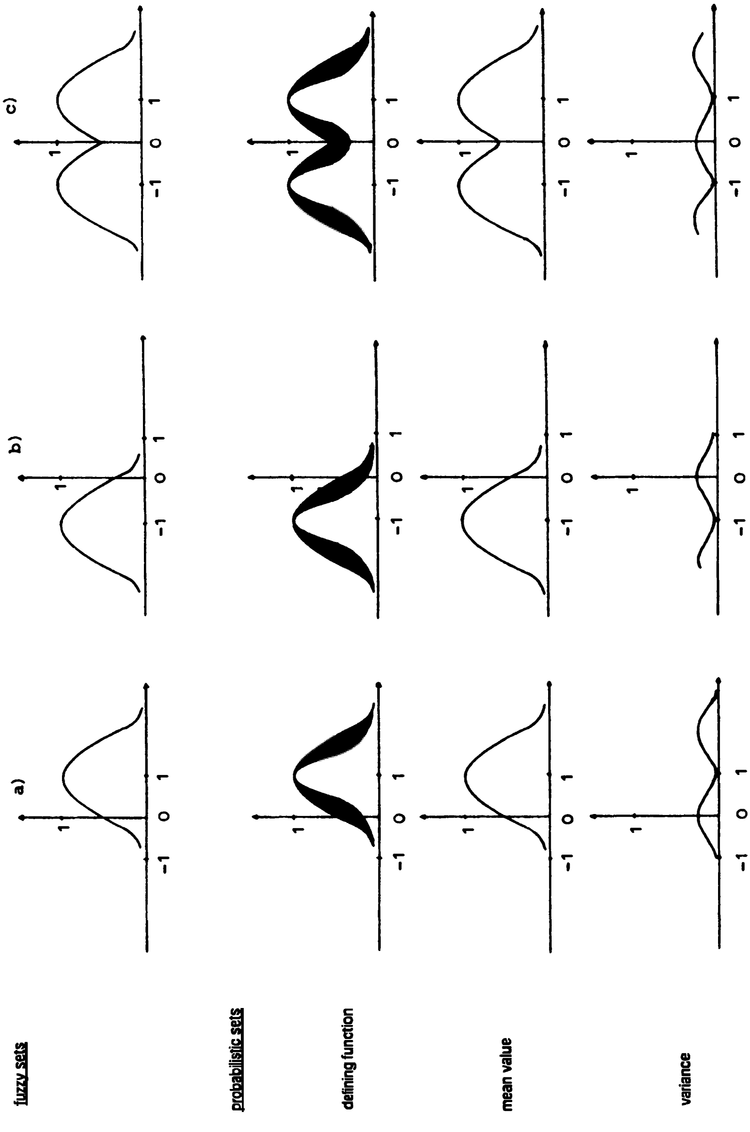


Figure 3–1. Fuzzy sets vs. probabilistic sets.

Ordinary fuzzy sets over X may be viewed as special intuitonistic fuzzy sets with the nonmembership function $\nu_A(x) = 1 - \mu_A(x)$. In the same way as fuzzy sets, intuitonistic L-fuzzy sets were defined by mapping the membership functions into a partially ordered set L [Atanassov and Stoeva 1984].

Definition 3–5 [Pawlak 1985, p. 99; Pawlak et al. 1988]

Let U denote a set of objects called universe and let $R \subset U \times U$ be an equivalence relation on U. The pair $A = (U, R)$ is called an approximation space. For u, $v \in U$ and $(u, v) \in R$, u and v belong to the same equivalence class, and we say that they are *indistinguishable* in A. Therefore the relation R is called an *indiscernibility* relation. Let $[x]_R$ denote an equivalence class (elementary set of A) of R containing element x; then lower and upper approximations for a subset $X \subseteq U$ in A—denoted $\underline{A}(X)$ and $\bar{A}(X)$, respectively—are defined as follows:

$$\underline{A}(X) = \{x \in U \| [x]_R \subset X\}$$
$$\bar{A}(X) = \{x \in U \| [x]_R \cap X \neq \theta\}$$

If an object x belongs to the lower approximation space of X in A, then "x surely belongs to X in A," $x \in \bar{A}(X)$ means that "x possibly belongs to X in A."

For the subset $X \subseteq U$ representing a concept of interest, the approximation space $A = (U, R)$ can be characterized by three distinct regions of X in A: the so-called positive region $\underline{A}(X)$, the boundary region $\bar{A}(X) - \underline{A}(X)$, and the negative region $U - \bar{A}(X)$.

The characterization of objects in X by the indiscernibility relation R is not precise enough if the boundary region $\bar{A}(X) - \underline{A}(X)$ is not empty. For this case it may be impossible to say whether an object belongs to X or not, and so the set X is said to be nondefinable in A, and X is a *rough set.*

Pawlak [1985] shows that the concept of approximation given by the equivalence relation R and the approximation space may not, in general, be replaced by a membership function similar to that introduced by Zadeh.

In order to take probabilistic informations crucial to nondeterministic classification problems into account, a natural probabilistic extension of the rough-set model has been proposed [Pawlak et al. 1988].

3.2 Further Operations on Fuzzy Sets

For the time being we return to ordinary fuzzy sets (type 1 fuzzy sets) and consider additional operations on them that have been defined in the literature and that will be useful or even necessary for later chapters.

3.2.1 Algebraic Operations

Definition 3–6

The *Cartesian product* of fuzzy sets is defined as follows: Let $\tilde{A}_1, \ldots, \tilde{A}_n$ be fuzzy sets in $X_1, \ldots, X_n$. The Cartesian product is then a fuzzy set in the product space $X_1 \times \ldots \times X_n$ with the membership function

$$\mu_{(\tilde{A}_1 x \ldots x \tilde{A}_n)}(x) = \min_i \{\mu_{\tilde{A}_i}(x_i) | x = (x_1, \ldots x_n), x_i \in X_i\}$$

Definition 3–7

The *mth power* of a fuzzy set $\tilde{A}$ is a fuzzy set with the membership function

$$\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m, \quad x \in X$$

Additional algebraic operations are defined as follows:

Definition 3–8

The *algebraic sum* (probabilistic sum) $\tilde{C} = \tilde{A} + \tilde{B}$ is defined as

$$\tilde{C} = \{(x, \mu_{\tilde{A}+\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$

Definition 3–9

The *bounded sum* $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined as

$$\tilde{C} = \{(x, \mu_{\tilde{A}\oplus\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\oplus\tilde{B}}(x) = \min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\}$$

Definition 3–10

The *bounded difference* $\tilde{C} = \tilde{A} \ominus \tilde{B}$ is defined as

$$\tilde{C} = \{(x, \mu_{\tilde{A}\ominus\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\ominus\tilde{B}}(x) = \max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\}$$

Definition 3–11

The *algebraic product* of two fuzzy sets $\tilde{C} = \tilde{A} \cdot \tilde{B}$ is defined as

$$\tilde{C} = \{(x, \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)) | x \in X\}$$

Example 3–1

Let $\tilde{A}(x) = \{(3, .5), (5, 1), (7, .6)\}$
$\tilde{B}(x) = \{(3, 1), (5, .6)\}$
The above definitions are then illustrated by the following results:

$$\begin{aligned}
\tilde{A} \times \tilde{B} &= \{[(3;3), .5], [(5;3), 1], [(7;3), .6] \\
&\qquad [(3;5), .5], [(5;5), .6], [(7;5), .6]\} \\
\tilde{A}^2 &= \{(3, .25), (5, 1), (7, .36)\} \\
\tilde{A} + \tilde{B} &= \{(3, 1), (5, 1), (7, .6)\} \\
\tilde{A} \oplus \tilde{B} &= \{(3, 1), (5, 1), (7, .6)\} \\
\tilde{A} \ominus \tilde{B} &= \{(3, .5), (5, .6)\} \\
\tilde{A} \cdot \tilde{B} &= \{(3, .5), (5, .6)\}
\end{aligned}$$

3.2.2 Set-Theoretic Operations

In chapter 2 the intersection of fuzzy sets, interpreted as the logical "and," was modeled as the min-operator and the union, interpreted as "or," as the max-operator. Other operators have also been suggested. These suggestions vary with respect to the generality or adaptibility of the operators as well as to the degree to which and how they are justified. Justification ranges from intuitive argumentation to empirical or axiomatic justification. Adaptability ranges from uniquely defined (for example, nonadaptable) concepts via parameterized "families" of operators to general classes of operators that satisfy certain properties.

We shall investigate the two basic classes of operators: operators for the intersection and union of fuzzy sets—referred to as triangular norms and conorms—and the class of averaging operators, which model connectives for fuzzy sets between t-norms and t-conorms. Each class contains parameterized as well as nonparameterized operators.

t-norms. T-norms were initiated in 1942 with the paper "Statistical metrics" [Menger 1942]. Menger intended to construct metric spaces where probability distributions rather than numbers are used in order to describe the distance between two elements in the respective space. Berthold Schweizer and Abe Sklar [Schweizer and Sklar 1961] provided the axioms of t-norms as they are used today.

The mathematical aspects of t-norms are excellently presented in the book by Klement, Mesiar and Pap [Klement et al. 2000]. The use of t-norms and t-conorms for modeling the intersection and union of fuzzy sets goes back to the 70s, see e.g. [Kruse et al. 1994]. Another source is basic psycho-linguistic research that tried to model quantitatively the linguistic "and" and "or" [Zimmermann and Zysno 1980, 1982, 1983, Thole, Zimmermann and Zysno 1979]. In the following we shall concentrate on those t-norms and t-conorms which are most common in fuzzy set theory. For mathematical derivations, proofs and other t-norms the reader is referred to the above-mentioned book by [Klement et al. 1994].

Let us first turn to basic definitions:

Definition 3–12 [Dubois and Prade 1980a, p. 17]

t-norms are two-valued functions from $[0, 1] \times [0, 1]$ that satisfy the following conditions:

1. $t(0, 0) = 0;\ t(\mu_{\tilde{A}}(x), 1) = t(1, \mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x), \quad x \in X$
2. $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \le t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \le \mu_{\tilde{D}}(x)$ (monotonicity)
3. $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$ (commutativity)
4. $t(\mu_{\tilde{A}}(x), t(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$ (associativity)

The functions t define a general class of intersection operators for fuzzy sets. The operators belonging to this class of t-norms are, in particular, associative (see condition 4), and therefore it is possible to compute the membership values for the intersection of more than two fuzzy sets by recursively applying a t-norm operator [Bonissone and Decker 1986, p. 220].

t-conorms (or s-norms). For the union of fuzzy sets, the max-operator, the algebraic sum [Zadeh 1965], and the "bold union" [Giles 1976]—modeled by the "bounded sum"—have been suggested.

Corresponding to the class of intersection operators, a general class of aggregation operators for the union of fuzzy sets called *triangular conorms* or *t-conorms* (sometimes referred to as *s-norms*) is defined analogously [Dubois and

Prade 1985, p. 90; Mizumoto 1989, p. 221]. The max-operator, algebraic sum, and bounded sum considered above belong to this class.

Definition 3–13 [Dubois and Prade 1985, p. 90]

t-conorms or *s*-norms are associative, commutative, and monotonic two-placed functions s that map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions:

1. $s(1, 1) = 1;\ s(\mu_{\tilde{A}}(x), 0) = s(0, \mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x), \quad x \in X$
2. $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \le s(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \le \mu_{\tilde{D}}(x)$ (monotonicity)
3. $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = s(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$ (commutativity)
4. $s(\mu_{\tilde{A}}(x), s(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = s(s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$ (associativity)

t-norms and t-conorms are related in a sense of logical duality. Alsina [Alsina 1985] defined a t-conorm as a two-placed function s mapping from $[0, 1] \times [0, 1]$ in $[0, 1]$ such that the function t, defined as

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = 1 - s(1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x))$$

is a t-norm. So any t-conorm s can be generated from a t-norm t through this transformation. More generally, Bonissone and Decker [1986] showed that for suitable negation operators like the complement operator for fuzzy sets—defined as $n(\mu_{\tilde{A}}(x)) = 1 - \mu_{\tilde{A}}(x)$ (see chapter 2)—pairs of t-norms t and t-conorms s satisfy the following generalization of DeMorgan's law [Bonissone and Decker 1986, p. 220]:

$$\begin{aligned} &s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = n(t(n(\mu_{\tilde{A}}(x)), n(\mu_{\tilde{B}}(x)))) \quad \text{and} \\ &t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = n(s(n(\mu_{\tilde{A}}(x)), n(\mu_{\tilde{B}}(x)))), \quad x \in X \end{aligned}$$

Typical dual pairs of nonparameterized t-norms and t-conorms are compiled below [Bonissone and Decker 1986, p. 221; Mizumoto 1989, p. 220]:

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{drastic product}$$

$$s_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = 0 \\ 1 & \text{otherwise} \end{cases} \qquad \text{drastic sum}$$

$$t_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\} \qquad \text{bounded difference}$$

$$s_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\}$$ bounded sum

$$t_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{2 - [\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)]}$$ Einstein product

$$s_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1 + \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}$$ Einstein sum

$$t_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$ algebraic product

$$s_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$ algebraic sum

$$t_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}$$ Hamacher product

$$s_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 2\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{1 - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}$$ Hamacher sum

$$t_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$ minimum

$$s_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$ maximum

These operators are ordered as follows:

$$t_w \le t_1 \le t_{1.5} \le t_2 \le t_{2.5} \le t_3$$
$$s_3 \le s_{2.5} \le s_2 \le s_{1.5} \le s_1 \le s_w$$

We notice that this order implies that for any fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X with membership values between 0 and 1, any intersection operator that is a t-norm is bounded by the min-operator and the operator t_w. A t-conorm is bounded by the max-operator and the operator s_w, respectively [Dubois and Prade 1982a, p. 42]:

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \le t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \le \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$
$$\max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \le s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \le s_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \quad x \in X$$

It may be desirable to extend the range of the previously described operators in order to adapt them to the context in which they are used. To this end, different authors suggested parameterized families of t-norms and t-conorms, often maintaining the associativity property.

For illustration purposes, we review some interesting parameterized operators. Some of these operators and their equivalence to the logical "and" and "or," respectively, have been justified axiomatically. We shall sketch the axioms on which the Hamacher-operator rests in order to give the reader the opportunity to compare the axiomatic system of Bellman and Giertz (min/max) on the one hand with that of the Hamacher-operator (which is essentially a family of product operators) on the other.

Definition 3–14 [Hamacher 1978]

The *intersection* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)}{\gamma + (1-\gamma)(\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x))}, \quad \gamma \geq 0$$

Hamacher wants to derive a mathematical model for the "and" operator. His basic axioms are as follows:

A1. The operator $\wedge$ is associative, that is, $\tilde{A} \wedge (\tilde{B} \wedge \tilde{C}) = (\tilde{A} \wedge \tilde{B}) \wedge \tilde{C}$.
A2. The operator $\wedge$ is continuous.
A3. The operator $\wedge$ is injective in each argument, that is,

$$(\tilde{A} \wedge \tilde{B}) = (\tilde{A} \wedge \tilde{C}) \Rightarrow \tilde{B} = \tilde{C}$$
$$(\tilde{A} \wedge \tilde{B}) = (\tilde{C} \wedge \tilde{B}) \Rightarrow \tilde{A} = \tilde{C}$$

(this is the essential difference between the Hamacher-operator and the Bellman–Giertz axioms).
A4. $\mu_{\tilde{A}}(x) = 1 \Rightarrow \mu_{\tilde{A}\wedge\tilde{A}}(x) = 1$

He then proves that a function f: $R \to [0, 1]$ exists with

$$\mu_{\tilde{A}\wedge\tilde{B}}(x) = f(f^{-1}(\mu_{\tilde{A}}(x)) + f^{-1}(\mu_{\tilde{B}}(x)))$$

If f is a rational function in $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$, then the only possible operator is that shown in definition 3–14. (For $\gamma = 1$, this reduces to the algebraic product!)

Notice that the Hamacher-operator is the only H-strict t-norm that can be expressed as a rational function [Mizumoto 1989, p. 223].

Definition 3–15 [Hamacher 1978]

The *union* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \cup \tilde{B} = \{(x, \mu_{\tilde{A}\cup\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \frac{(\gamma' - 1)\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x) + \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1 + \gamma'\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)}, \quad \gamma' \geq -1$$

For $\gamma' = 0$ the Hamacher-union-operator reduces to the algebraic sum.

Yager [1980] defined another triangular family of operators.

Definition 3–16 [Yager 1980]

The *intersection* of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = 1 - \min\left\{1, \left((1 - \mu_{\tilde{A}}(x))^p + (1 - \mu_{\tilde{B}}(x))^p\right)^{1/p}\right\}, \quad p \geq 1$$

The *union* of fuzzy sets is defined as

$$\tilde{A} \cup \tilde{B} = \{(x, \mu_{\tilde{A}\cup\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \min\left\{1, \left(\mu_{\tilde{A}}(x)^p + \mu_{\tilde{B}}(x)^p\right)^{1/p}\right\}, \quad p \geq 1$$

His intersection-operator converges to the min-operator for $p \to \infty$ and his union operator to the max-operator for $p \to \infty$.

For $p = 1$ the Yager-intersection becomes the "bold-intersection" of definition 3–10. The union operator converges to the maximum-operator for $p \to \infty$ and to the bold union for $p = 1$. Both operators satisfy the DeMorgan laws and are commutative, associative for all p, and monotonically nondecreasing in $\mu(x)$; they also include the classical cases of dual logic. They are, however, not distributive.

Dubois and Prade [1980c, 1982a] also proposed a commutative and associative parameterized family of aggregation operators:

Definition 3–17 [Dubois and Prade 1980c, 1982a]

The *intersection* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}\cap\tilde{B}}(x)=\frac{\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{\max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x),\alpha\}},\quad \alpha\in[0,1]$$

This intersection-operator is decreasing with respect to α and lies between min $\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)\}$ (which is the resulting operation for $\alpha = 0$) and the algebraic product $\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)$ (for $\alpha = 1$). The parameter α is a kind of threshold, since the following relationships hold for the defined intersection operation [Dubois and Prade 1982a, p. 47]:

$$\mu_{\tilde{A}\cap\tilde{B}}(x)=\min\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}\quad\text{for}\quad \mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\in[\alpha,1]$$

$$\mu_{\tilde{A}\cap\tilde{B}}(x)=\frac{\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{\alpha}\quad\text{for}\quad \mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\in[0,\alpha]$$

Definition 3–18 [Dubois and Prade 1980c, 1982a]

For the *union* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$, defined as

$$\tilde{A}\cup\tilde{B}=\{(x,\mu_{\tilde{A}\cup\tilde{B}}(x))|x\in X\}$$

Dubois and Prade suggested the following operation, where $\alpha \in [0, 1]$:

$$\mu_{\tilde{A}\cup\tilde{B}}(x)=\frac{\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)-\min\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x),(1-\alpha)\}}{\max\{(1-\mu_{\tilde{A}}(x)),(1-\mu_{\tilde{B}}(x)),\alpha\}}$$

All the operators mentioned so far include the case of dual logic as a special case. The question may arise: Why are there unique definitions for intersection (= and) and union (= or) in dual logic and traditional set theory and so many suggested definitions in fuzzy set theory? The answer is simply that many operators (for instance, product and min-operator) perform in exactly the same way if the degrees of membership are restricted to the values 0 or 1. If this restriction is no longer required, the operators lead to different results.

This triggers yet another question: Are the only ways to "combine" or aggregate fuzzy sets the intersection or union—or the logical "and" or "or"—respectively? Or are there other possibilities of aggregation? The answer to this latter question is definitely yes. There are other ways of combining fuzzy sets and fuzzy statements; "and" and "or" are only limiting special cases. Generalized models for the logical "and" and "or" are given by the "fuzzy and" and "fuzzy or" [Werners 1984]. Furthermore, a number of authors have suggested general connectives, which are (so far) of particular importance for decision analysis and for other applications of fuzzy set theory. These operators are general in the sense that they do not distinguish between the intersection and union of fuzzy sets.

Here we shall only mention some of these general connectives. A detailed discussion of them and the description of still others can be found in volume 2 in the context of decision making in fuzzy environments.

Averaging Operators. A straightforward approach for aggregating fuzzy sets (for instance, in the context of decision making) would be to use the aggregating procedures frequently used in utility theory or multicriteria decision theory. These procedures realize the idea of trade-offs between conflicting goals when compensation is allowed, and the resulting trade-offs lie between the most optimistic lower bound and the most pessimistic upper bound, that is, they map between the minimum and the maximum degree of membership of the aggregated sets. Therefore they are called averaging operators. Operators such as the weighted and unweighted arithmetic or geometric mean are examples of nonparametric averaging operators. In fact, they are adequate models for human aggregation procedures in decision environments and have empirically performed quite well [Thole, Zimmermann, and Zysno 1979]. Procedures and results of empirical research done in the context of human decision making are investigated in section 14.3.

The fuzzy aggregation operators "fuzzy and" and "fuzzy or" suggested by Werners [1984] combine the minimum and maximum operator, respectively, with the arithmetic mean. The combination of these operators leads to very good results with respect to empirical data [Zimmermann and Zysno 1983] and allows compensation between the membership values of the aggregated sets.

Definition 3–19 [Werners 1988, p. 297]

The "fuzzy and" operator is defined as

$$\mu_{\widetilde{\text{and}}}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \gamma \cdot \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \frac{(1-\gamma)(\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x))}{2}$$

$$x \in X, \gamma \in [0, 1]$$

The "fuzzy or" operator is defined as

$$\mu_{\widetilde{\text{or}}}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \gamma \cdot \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \frac{(1-\gamma)(\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x))}{2}$$

$$x \in X, \gamma \in [0, 1]$$

The parameter γ indicates the degree of nearness to the strict logical meaning of "and" and "or," respectively. For $\gamma = 1$, the "fuzzy and" becomes the minimum operator, and the "fuzzy or" reduces to the maximum operator. $\gamma = 0$ yields for both the arithmetic mean.

Additional averaging aggregation procedures are symmetric summation operators, which, like the arithmetic or geometric mean operators, indicate some degree of compensation but in contrast to the latter are not associative. Examples of symmetric summation operators are the operators M_1, M_2, and N_1, N_2, known as symmetric summations and symmetric differences, respectively. Here the aggregation of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is pointwise defined as follows:

$$M_1(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x))=\frac{\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{1+\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-2\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}$$

$$M_2(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x))=\frac{\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{1+\mu_{\tilde{A}}(x)-\mu_{\tilde{B}}(x)+2\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}$$

$$N_1(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x))=\frac{\max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}}{1+|\mu_{\tilde{A}}(x)-\mu_{\tilde{B}}(x)|}$$

$$N_2(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x))=\frac{\min\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}}{1+|\mu_{\tilde{A}}(x)-\mu_{\tilde{B}}(x)|}$$

A detailed description of the properties of nonparametric averaging operators is reported by Dubois and Prade [1984]. For further details of symmetric summation operators, the reader is referred to Silvert [1979].

The above-mentioned averaging operators indicate a "fix" compensation between the logical "and" and the logical "or." In order to describe a variety of phenomena in decision situations, several operators with different compensations are necessary. An operator that is more general in the sense that the compensation between intersection and union is expressed by a parameter γ was suggested and empirically tested by Zimmermann and Zysno [1980] under the name "compensatory and."

Definition 3–20 [Zimmermann and Zysno 1980]

The *"compensatory and" operator* is defined as follows:

$$\mu_{\tilde{A}i,\text{comp}}(x)=\left(\prod_{i=1}^{m}\mu_i(x)\right)^{(1-\gamma)}\left(1-\prod_{i=1}^{m}(1-\mu_i(x))\right)^{\gamma},\quad x\in X, 0\le\gamma\le 1$$

This "γ-operator" is obviously a combination of the algebraic product (modeling the logical "and") and the algebraic sum (modeling the "or"). It is pointwise injective (except at zero and one), continuous, monotonous, and commutative. It also satisfies the DeMorgan laws and is in accordance with the truth tables of dual logic. The parameter indicates where the actual operator is located between the logical "and" and "or."

Other operators following the idea of parameterized compensation are defined by taking linear convex combinations of noncompensatory operators modeling the logical "and" and "or." The aggregation of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ by the convex combination between the min- and max-operator is defined as

$$\mu_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \gamma \cdot \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + (1-\gamma) \cdot \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \qquad \gamma \in [0, 1]$$

Combining the algebraic product and algebraic sum, we obtain the following operation:

$$\mu_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \gamma \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x) + (1-\gamma) \cdot [\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)] \qquad \gamma \in [0, 1]$$

This class of operators is again in accordance with the dual logic truth tables. But Zimmermann and Zysno showed that the "compensatory and" operator is more adequate in human decision making than are these operators [Zimmermann and Zysno 1980, p. 50].

The relationships between different aggregation operators for aggregating two fuzzy sets $\tilde{A}$ and $\tilde{B}$ with respect to the three classes of t-norms, t-conorms, and averaging operators are represented in figure 3–2.

A taxonomy with respect to the compensatory property of distinguishing operators, which differentiate between the intersection and union of fuzzy sets, and general operators is presented in table 3–1. Table 3–2 summarizes the classes of aggregation operators for fuzzy sets reported in this chapter and compiles some references. Table 3–3 represents the relationship between parameterized families

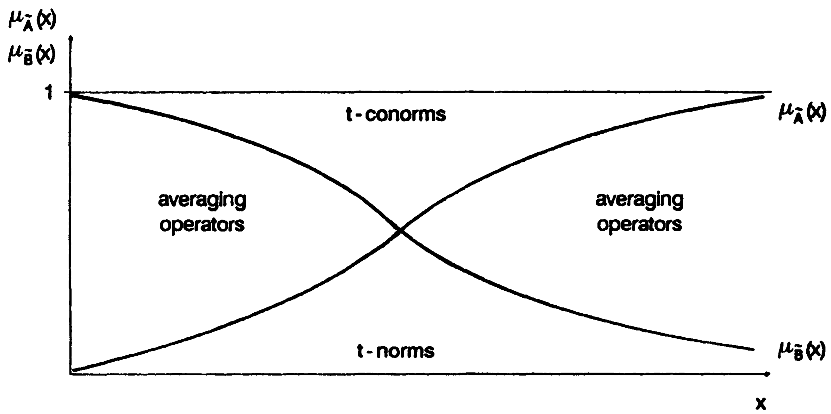


Figure 3–2. Mapping of t-norms, t-conorms, and averaging operators.

Table 3–1. Classification of compensatory and noncompensatory operators.

	Distinguishing operators	*General operators*
Compensatory	fuzzy and fuzzy or	compensatory and convex combinations of min and max symmetric summations mean operators
Noncompensatory	*t*-norms *t*-conorms min max	

of operators and the presented *t*-norms and *t*-conorms with respect to special values of their parameters.

Ordered Weighted Averaging (OWA) Operators. Yager [Yager 1988] introduced a family of aggregation operators, so-called OWA operators, which belong into the class of mean operators. They are especially suited—and intended—to aggregations of (weighted) criteria in multi criteria decision making that will be discussed in chapter 13. Yager uses the same idea that is behind definition 3–20, i.e. that for the aggregation of criteria an "operator" between the "logical and" and the "logical inclusive or" seems to be suitable. By contrast to the "compensatory and", defined in 3–20, Yager derives his suggestions by formal arguments rather than by scientific empirical tests:

Definition 3–21 [Yager 1993]

An OWA-operator is defined as follows:

$$\mu_{OWA}(x) = \sum_j w_j \mu_j(x)$$

where: $w = \{w_1, w_2, \ldots w_n\}$ is a vector of weights w_i with

$$w_i \in [0,1] \quad \text{and}$$
$$\sum_i w_i = 1$$

$\mu_j(x)$ is the j^{th} largest membership value for an element x for which the (aggregated) degree of membership shall be determined.

The rationale behind this operator is again the observation, that for an "and" aggregation (modeled i.e. by the min-operator) the smallest degree of

Table 3–2. Classification of aggregation operators.

References	*Intersection operators* *t-norms*	*Averaging operators*	*Union operators* *t-conorms*
		Nonparameterized	
Zadeh 1965	minimum algebraic product		maximum algebraic sum
Giles 1976	bounded sum		bounded difference
Hamacher 1978	Hamacher product		Hamacher sum
Mizumoto 1982	Einstein product		Einstein sum
Dubois and Prade 1980, 1982	drastic product		drastic sum
Dubois and Prade 1984		arithmetic mean geometric mean	
Silvert 1979		symmetric summation and differences	
		Parameterized families	
Hamacher 1978	Hamacher-intersection-operators		Hamacher-union-operators
Yager 1980	Yager-intersection-operators		Yager-union-operators
Dubois and Prade 1980a, 1982a, 1984	Dubois-intersection-operators		Dubois-union-operators
Werners 1984		“fuzzy and”, “fuzzy or”	
Zimmermann and Zysno 1980		“compensatory and”, convex comb. of maximum and minimum, or algebraic product and algebraic sum	
Yager 1988		OWA-operators	

Table 3–3. Relationship between parameterized operators, their parameters, and other t/s-norms.

	Types of t-norms and t-conorms											
	drastic		*bounded*		*Einstein*		*algebraic*		*Hamacher*			
Parameterized operators	*prod.*	*sum*	*sum*	*diff.*	*prod.*	*sum*	*prod.*	*sum*	*prod.*	*sum*	*min*	*max*
Hamacher												
intersection	$\gamma \to \infty$				$\gamma = 2$		$\gamma = 1$		$\gamma = 0$			
union		$\gamma' \to \infty$				$\gamma' = 1$		$\gamma' = 0$		$\gamma' = -1$		
Yager												
intersection	$p \to 0$		$p = 1$								$p \to \infty$	
union		$p \to 0$		$p = 1$								$p \to \infty$
Dubois												
intersection							$\alpha = 1$				$\alpha = 0$	
union								$\alpha = 1$				$\alpha = 0$

membership is crucial while for an "or" aggregation (modeled by "max") the largest degree of membership of an element in all fuzzy sets is to be aggregated.

Therefore, a basic aspect of this operator is the re-ordering step. In particular, the degree of membership of an element in a fuzzy set is not associated with a particular weight.

Rather a weight is associated with a particular ordered position of a degree of membership in the ordered set of relevant degrees of membership.

Example 3–2

Let us consider the aggregation of the degrees of membership of an element x, which is contained in 10 fuzzy sets $\mu_1(x)$ to $\mu_{10}(x)$.

The OWA-weighting vector be:

$$w = (0.3, 0.20, 0.15, 0.12, 0.06, 0.05, 0.04, 0.03, 0.02, 0.02)$$

The degrees of membership of x in the 10 fuzzy sets $\mu_i(x)$ are:

$$u = (0.2, 0.3, 0.5, 0.8, 1, 0.6, 0.5, 0.4, 0.3, 0.2)$$

Recording the $\mu_i(x)$ according to their values yields:

$$u' = (1, 0.8, 0.6, 0.5, 0.5, 0.4, 0.3, 0.3, 0.2, 0.2)$$

$$\begin{aligned}\mu_{OWA}(x) &= (0.3)(1) + (0.2)(0.8) + (0.15)(0.6) + (0.12)(0.5) + (0.06)(0.5) \\ &+ (0.05)(0.4) + (0.04)(0.3) + (0.03)(0.03) + (0.02)(0.2) + (0.02)(0.2) \\ &= 0.689\end{aligned}$$

Special vectors w correspond to typical aggregation operators. For instance:

$$w = (1, 0, 0, \ldots, 0) = \text{max-operator}$$

$$w = (0, 0, \ldots, 0, 1) = \text{min-operator}$$

$$w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = \frac{1}{n}\sum_i \mu_i(x) = \text{arith. mean}$$

Yager also defines a number of measures, two of which quantify the position of this operator between the "logical and" and the "logical or":

He defines the "orness" as

$$\text{orness}(w) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$$

and "andness" as

$$\text{andness}(w) = 1 - \text{orness}(w)$$

and suggests that values of less than .5 for these measures indicate a bias to "and" or "or" respectively (compare with definition 3–19!).

Yager also proposes other families of OWA-operators in [Yager 1993].

Finally it shall be mentioned that some authors also suggest the fuzzy integral as aggregation operator [Grabisch 1998].

3.2.3 Criteria for Selecting Appropriate Aggregation Operators

The variety of operators for the aggregation of fuzzy sets might be confusing and might make it difficult to decide which one to use in a specific model or situation. Which rules can be used for such a decision?

The following eight important criteria according to which operators can be classified are not quite disjunct; hopefully they may be helpful in selecting the appropriate connective.

1. **Axiomatic Strength.** We have listed the axioms that Bellman-Giertz and Hamacher, respectively, wanted their operators to satisfy. Obviously, everything else being equal, an operator is the better the less limiting the axioms are it satisfies.
2. **Empirical Fit.** If fuzzy set theory is used as a modeling language for real situations or systems, it is not only important that the operators satisfy certain axioms or have certain formal qualities (such as associativity, commutativity), which are certainly of importance from a mathematical point of view, but also the operators must be appropriate models of real-system behavior; and this can normally be proven only by empirical testing.
3. **Adaptability.** It is rather unlikely that the type of aggregation is independent of the context and semantic interpretation, that is, whether the aggregation of fuzzy sets models a human decision, a fuzzy controller, a medical diagnostic system, or a specific inference rule in fuzzy logic. If one wants to use a very small number of operators to model many situations, then these operators have to be adaptable to the specific context. This can, for instance, be achieved by parameterization. Thus min- and max-operators cannot be adapted at all. They are acceptable in situations in which they fit and under no other circumstances. (Of course, they have other advantages, such as numerical efficiency.) By contrast, Yager's operators or the γ-operator can be adapted to certain contexts by setting the p's or γ's appropriately, and OWA operators by chaosing appropriate weight vectors.

4. **Computational Efficiency.** If one compares the min-operator with, for instance, Yager's intersection operator or the γ-operator, it becomes quite obvious that the latter two require considerably more computational effort than the former. In practice, this might be quite important, in particular when large problems have to be solved.
5. **Compensation.** The logical "and" does not allow for compensation at all; that is, an element of the intersection of two sets cannot compensate for a low degree of belonging to one of the intersected sets by a higher degree of belonging to another of them. In (dual) logic, one cannot compensate by the higher truth of one statement for the lower truth of another statement when combining them by "and." By compensation, in the context of aggregation operators for fuzzy sets, we mean the following: Given that the degree of membership to the aggregated fuzzy set is

$$\mu_{\text{Agg}}(x_k) = f(\mu_{\tilde{A}}(x_k), \mu_{\tilde{B}}(x_k)) = k$$

f is compensatory if $\mu_{\text{Agg}}(x_k) = k$ is obtainable for a different $\mu_{\tilde{A}}(x_k)$ by a change in $\mu_{\tilde{B}}(x_k)$. Thus the min-operator is not compensatory, while the product operator, the γ-operator, and so forth, are.
6. **Range of Compensation.** If one would use a convex combination of min- and max-operator, a compensation could obviously occur in the range between min and max. The product operator allows compensation in the open interval (0, 1). In general, the larger the range of compensation, the better the compensatory operator.
7. **Aggregating Behavior.** If one considers normal or subnormal fuzzy sets, the degree of membership in the aggregated set depends very frequently on the number of sets combined. If one combines fuzzy sets by the product operator, for instance, each additional fuzzy set "added" will normally decrease the resulting aggregate degrees of membership. This might be a desirable feature; it might, however, also be inadequate. Goguen, for instance, argues that for formal reasons the resulting degree of membership should be nonincreasing [Goguen 1967].
8. **Required Scale Level of Membership Functions.** The scale level (nominal, interval, ratio, or absolute) on which membership information can be obtained depends on a number of factors. Different operators may require different scale levels of membership information to be admissible. (For instance, the min-operator is still admissible for ordinal information, while the product operator, strictly speaking, is not!) In general, again all else being equal, the operator that requires the lowest scale level is the most preferable from the point of view of information gathering.

Exercises

1. The product and the bounded difference have both been suggested as models for the intersection. Compute the intersection of fuzzy sets $\tilde{B}$ and $\tilde{C}$ from exercise 4 of chapter 2 and compare the three alternative models for the intersection: Minimum, product, and bounded difference.
2. The bounded sum and the algebraic sum have been suggested as alternative models for the union of fuzzy sets. Compute the union of the fuzzy sets $\tilde{B}$ and $\tilde{C}$ of exercise 4 of chapter 2 using the above-mentioned models, and compare the result with the result of exercise 4 of chapter 2.
3. Determine the intersection of $\tilde{B}$ and $\tilde{C}$ in exercise 4 of chapter 2 by using the
   a. Hamacher operator with $\gamma = .25; .5; .75$
   b. Yager operator with $p = 1, 5, 10$.
4. Which of the intersection operators mentioned in chapter 3 are compensatory and which not? Are the "compensatory" operators compensatory for the entire range [0, 1] and for the entire domain of their parameters (γ, p, etc.)? If not, what are the limits of compensation?
5. Prove that the following properties are satisfied by Yager's union operator:
   a. $\mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{B}}(x) = 0$
   b. $\mu_{\tilde{A} \cup \tilde{B}}(x) = 1$ for $\mu_{\tilde{B}}(x) = 1$
   c. $\mu_{\tilde{A} \cup \tilde{B}}(x) \geq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$
   d. For $p \to 0$, the Yager union operator reduces to s_w (drastic sum).
6. Show for the parameterized families of fuzzy union defined by Hamacher, Yager, and Dubois that the defining functions of these operators decrease with any increase in the parameter.

4 FUZZY MEASURES AND MEASURES OF FUZZINESS

4.1 Fuzzy Measures

In order to prevent confusion about fuzzy measures and measures of fuzziness, we shall first briefly describe the meaning and features of fuzzy measures. In the late 1970s, Sugeno defined a fuzzy measure as follows:

Sugeno [1977]: $\mathcal{B}$ is a Borel field of the arbitrary set (universe) X.

Definition 4–1

A set function g defined on $\mathcal{B}$ that has the following properties is called a *fuzzy measure*:

1. $g(0) = 0$, $g(X) = 1$.
2. If $A, B \in \mathcal{B}$ and $A \subseteq B$, then $g(\mathrm{A}) \leq g(\mathrm{B})$.
3. If $A_n \in \mathcal{B}$, $A_1 \subseteq A_2 \subseteq \ldots$, then $\lim\limits_{n\to\infty} g(A_n) = g(\lim\limits_{n\to\infty} A_n)$.

Sugeno's measure differs from the classical measure essentially by relaxing the additivity property [Murofushi and Sugeno 1989, p. 201]. A different approach, however, is used by Klement and Schwyhla [1982]. The interested reader is referred to their article.

Banon [1981] shows that very many measures with finite universe, such as probability measures, belief functions, plausibility measures, and so on, are fuzzy measures in the sense of Sugeno. For this book, one measure—possibility—is of particular interest [see Dubois and Prade 1988a, p. 7].

In the framework of fuzzy set theory, Zadeh introduced the notion of a possibility distribution and the concept of a possibility measure, which is a special type of the fuzzy measure proposed by Sugeno. A possibility measure is defined as follows [Zadeh 1978; Higashi and Klir 1982]:

Definition 4–2

Let $P(X)$ be the power set of a set X.
A *possibility measure* is a function Π: $P(X) \to [0, 1]$ with the properties

1. $\Pi(0) = 0$, $\Pi(X) = 1$
2. $A \subseteq B \Rightarrow \Pi(A) \le \Pi(B)$
3. $\Pi(\bigcup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i)$ with an index set I.

It can be uniquely determined by a possibility distribution function f: $X \to [0, 1]$ by $\Pi(A) = \sup_{x \in A} f(x)$, $A \subset X$. It follows directly that f is defined by $f(x) = \Pi(\{x\}) \forall_x \in X$ [Klir and Folger 1988, p. 122].

A possibility is not always a fuzzy measure [Puri and Ralescu 1982]. It is, however, a fuzzy measure if X is finite and if the possibility distribution is normal—that is, a mapping into [0, 1].

Example 4–1

Let $X = \{0, 1, \ldots, 10\}$.
$\Pi(\{x\})$: = Possibility that x is close to 8.

x	0	1	2	3	4	5	6	7	8	9	10
$\Pi(\{x\})$	.0	.0	.0	.0	.0	.1	.5	.8	1	.8	.5

$\Pi(A)$: = Possibility that A contains an integer close to 8.

$$A \subset X \Rightarrow \Pi(A) = \sup_{x \in A} \Pi(\{x\})$$

For $A = \{2, 5, 9\}$ we compute:

$$\begin{aligned}\Pi(A) &= \sup_{x \in A} \Pi(\{x\}) \\ &= \sup\{\Pi(\{2\}), \Pi(\{5\}), \Pi(\{9\})\} \\ &= \sup\{0, .1, .8\} \\ &= .8\end{aligned}$$

4.2 Measures of Fuzziness

Measures of fuzziness, in contrast to fuzzy measures, try to indicate the degree of fuzziness of a fuzzy set. A number of approaches to this end have become known. Some authors, strongly influenced by the Shannon entropy as a measure of information, and following de Luca and Termini [1972], consider a measure of fuzziness as a mapping d from the power set $P(X)$ to $[0, +\infty]$ that satisfies a number of conditions. Others [Kaufmann 1975] suggested an index of fuzziness as a normalized distance, and others [Yager 1979; Higashi and Klir 1982] base their concept of a measure of fuzziness on the degree of distinction between the fuzzy set and its complement.

We shall, as an illustration, discuss two of those measures. Suppose for both cases that the support of A is finite.

The first is as follows: Let $\mu_{\tilde{A}}(x)$ be the membership function of the fuzzy set $\tilde{A}$ for $x \in X$, X finite. It seems plausible that the measure of fuzziness $d(\tilde{A})$ should then have the following properties [de Luca and Termini 1972]:

1. $d(\tilde{A}) = 0$ if $\tilde{A}$ is a crisp set in X.
2. $d(\tilde{A})$ assumes a unique maximum if $\mu_{\tilde{A}}(x) = \frac{1}{2} \forall x \in X$.
3. $d(\tilde{A}) \geq d(\tilde{A}')$ if $\tilde{A}'$ is "crisper" than $\tilde{A}$, i.e., if $\mu_{\tilde{A}'}(x) \leq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) \leq \frac{1}{2}$ and $\mu_{\tilde{A}'}(x) \geq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) \geq \frac{1}{2}$.
4. $d(\complement\tilde{A}) = d(\tilde{A})$ where $\complement\tilde{A}$ is the complement of $\tilde{A}$.

De Luca and Termini suggested as a measure of fuzziness the "entropy"[1] of a fuzzy set [de Luca and Termini 1972, p. 305], which they defined as follows:

Definition 4–3a

The entropy as a *measure of a fuzzy set* $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$ is defined as

[1] Also employed in thermodynamics, information theory, and statistics [Capocelli and de Luca 1973].

$$d(\tilde{A}) = H(\tilde{A}) + H(\complement\tilde{A}), \quad x \in X$$

$$H(\tilde{A}) = -K\sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)\ln(\mu_{\tilde{A}}(x_i))$$

where n is the number of elements in the support of $\tilde{A}$ and K is a positive constant.

Using Shannon's function $S(x) = -x\ln x - (1-x)\ln(1-x)$, de Luca and Termini simplify the expression in definition 4–3a to arrive at the following definition.

Definition 4–3b

The entropy d as a *measure of fuzziness* of a fuzzy set $\tilde{A} = \{x, \mu_{\tilde{A}}(x)\}$ is defined as

$$d(\tilde{A}) = K\sum_{i=1}^{n} S(\mu_{\tilde{A}}(x_i)).$$

Example 4–2

Let $\tilde{A}$ = "integers close to 10" (see example 2–1d)

$$\tilde{A} = \{(7, .1), (8, .5), (9, .8), (10, 1), (11, .8), (12, .5), (13, .1)\}$$

Let $K = 1$, so

$$d(\tilde{A}) = .325 + .693 + .501 + 0 + .501 + .693 + .611 + .325 = 3.038$$

Furthermore, let $\tilde{B}$ = "integers quite close to 10"

$$\tilde{B} = \{(6, .1), (7, .3), (8, .4), (9, .7), (10, 1), (11, .8), (12, .5), (13, .3), (14, .1)\}$$

$$d(\tilde{B}) = .325 + .611 + .673 + .611 + 0 + .501 + .693 + .611 + .325 = 4.35$$

The second measure is as follows: Knopfmacher [1975], Loo [1977], Gottwald [1979b], and others based their contributions on the Luca and Termini's suggestion in some respects.

If $\tilde{A}$ is a fuzzy set in X and $\complement\tilde{A}$ is its complement, then in contrast to crisp sets, it is not necessarily true that

$$\tilde{A} \cup \complement\tilde{A} = X$$

$$\tilde{A} \cap \complement\tilde{A} = \emptyset$$

This means that fuzzy sets do not always satisfy the law of the excluded middle, which is one of their major distinctions from traditional crisp sets. Some authors [Yager 1979; Higashi and Klir 1982] consider the relationship between $\tilde{A}$ and $\complement\tilde{A}$ to be the essence of fuzziness.

Yager [1979] notes that the requirement of distinction between $\tilde{A}$ and $\complement\tilde{A}$ is not satisfied by fuzzy sets. He therefore suggests that any measure of fuzziness should be a measure of the lack of distnction between $\tilde{A}$ and $\complement\tilde{A}$ or $\mu_{\tilde{A}}(x)$ and $\mu_{\complement\tilde{A}}(x)$.

As a possible metric to measure the *distance* between a fuzzy set and its complement, Yager suggests:

Definition 4–4

$$D_p(\tilde{A}, \complement\tilde{A}) = \left[\sum_{i=1}^{n} |\mu_{\tilde{A}}(x_i) - \mu_{\complement\tilde{A}}(x_i)|^p\right]^{1/p} \quad p = 1, 2, 3, \ldots$$

Let $S = \text{supp}(\tilde{A})$: $D_p(S, \complement S) = \|S\|^{1/p}$

Definition 4–5 [Yager 1979]

A *measure of the fuzziness* of $\tilde{A}$ can be defined as

$$f_p(\tilde{A}) = 1 - \frac{D_p(\tilde{A}, \complement\tilde{A})}{\|\text{supp}(\tilde{A})\|}$$

So $f_p(\tilde{A}) \in [0, 1]$. This measure also satisfies properties 1 to 4 required by de Luca and Termini (see above).

For $p = 1$, $D_p(\tilde{A}, \complement\tilde{A})$ yields the Hamming metric

$$D_1(\tilde{A}, \complement\tilde{A}) = \sum_{i=1}^{n} |\mu_{\tilde{A}}(x_i) - \mu_{\complement\tilde{A}}(x_i)|$$

Because $\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$, this becomes

$$D_1(\tilde{A}, \complement\tilde{A}) = \sum_{i=1}^{n} |2\mu_{\tilde{A}}(x_i) - 1|$$

For $p = 2$, we arrive at the Euclidean metric

$$D_2(\tilde{A}, \complement\tilde{A}) = \left(\sum_{i=1}^{n} (\mu_{\tilde{A}}(x_i) - \mu_{\text{C}\tilde{A}}(x_i))^2\right)^{1/2}$$

and for $\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$, we have

$$D_2(\tilde{A}, \complement\tilde{A}) = \left(\sum_{i=1}^{n} (2\mu_{\tilde{A}}(x_i) - 1)^2\right)^{1/2}$$

Example 4–3

Let $\tilde{A}$ = "integers close to 10" and
$\tilde{B}$ = "integers quite close to 10" be defined as in example 4–2.
Applying the above derived formula, we compute for $p = 1$:

$$D_1(\tilde{A}, \complement\tilde{A}) = .8 + 0 + .6 + 1 + .6 + 0 + .8$$
$$= 3.8$$
$$\|\text{supp}(\tilde{A})\| = 7$$

so $f_1(\tilde{A}) = 1 - \frac{3.8}{7} = 0.457.$

Analogously,

$$D_1(\tilde{B}, \complement\tilde{B}) = 4.6$$
$$\|\text{supp}(\tilde{B})\| = 9$$

so $f_1(\tilde{B}) = 1 - \frac{4.6}{9} = 0.489.$

Similarly, for $p = 2$, we obtain

$$D_2(\tilde{A}, \complement\tilde{A}) = 1.73$$
$$\|\text{supp}(\tilde{A})\| = 2.65$$

so $f_2(\tilde{A}) = 1 - \frac{1.73}{2.65} = 0.347,$ and

$$D_2(\tilde{B}, \complement\tilde{B}) = 1.78$$
$$\|\text{supp}(\tilde{B})\| = 1$$

so $f_2(\tilde{B}) = 1 - \frac{1.78}{3} = 0.407.$

The reader should realize that the complement of a fuzzy set is not uniquely defined [see Bellman and Giertz 1973; Dubois and Prade 1982a; Lowen 1978]. It is therefore not surprising that for other definitions of the complement and for other measures of distance, other measures of fuzziness will result, even though they all focus on the distinction between a fuzzy set and its complement [see, for example, Klir 1987, p. 141]. Those variations, as well as extension of measures of fuzziness to nonfinite supports, will not be considered here; neither will the approaches that define fuzzy measures of fuzzy sets [Yager 1979].

Exercises

1. Let $\tilde{A}$ be defined as in example 4–2.
   $\tilde{B}' = \{(8, .5), (9, .9), (10, 1), (11, .8), (12, .5)\}$
   $\tilde{C}' = \{(6, .1), (7, .1), (8, .5), (9, .8), (10, 1), (11, .8), (12, .5), (13, .1), (14, 1)\}$
   Is $\tilde{A}$ crisper than $\tilde{B}$ (or $\tilde{C}$)?
   Compute as measures of fuzziness:
   a. the entropy (with $K = 1$)
   b. f_1
   c. f_2 for all three sets.
   Compare the results.
2. Determine the maximum of the entropy of $d(\tilde{A})$ in dependence of the cardinality of the support of $\tilde{A}$.
3. Consider $\tilde{A}$ as in exercise 1. Determine $\tilde{A} \cap ¢\tilde{A}$ and $\tilde{A} \cup ¢\tilde{A}$. For which (special) fuzzy sets does the equality hold?
4. Consider example 4–1. Compute the possibilities of the following sets:

$$A_1 = \{1, 2, 3, 4, 5, 6\}, A_2 = \{1, 5, 8, 9\}, A_3 = \{7, 9\}$$

5 THE EXTENSION PRINCIPLE AND APPLICATIONS

5.1 The Extension Principle

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In its elementary form, it was already implied in Zadeh's first contribution [1965]. In the meantime, modifications have been suggested [Zadeh 1973a; Zadeh et al. 1975; Jain 1976]. Following Zadeh [1973a] and Dubois and Prade [1980a], we define the extension principle as follows:

Definition 5–1

Lex X be a Cartesian product of universes $X = X_1 x \ldots x X_r$, and $\tilde{A}_1, \ldots, \tilde{A}_r$ be r fuzzy sets in $X_1, \ldots, X_r$, respectively. f is a mapping from X to a universe Y, $y = f(x_1, \ldots, x_r)$. Then the extension principle allows us to define a *fuzzy set $\tilde{B}$ in Y* by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_r)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_r}(x_r)\} & \text{if} \quad f^{-1}(y) \neq \theta \\ 0 & \text{otherwise} \end{cases}$$

where f^{-1} is the inverse of f.

For $r = 1$, the extension principle, of course, reduces to

$$\tilde{B} = f(\tilde{A}) = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in X\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if} \quad f^{-1}(y) \neq \theta \\ 0 & \text{otherwise} \end{cases}$$

Example 5–1

Let $\tilde{A} = \{(-1, .5), (0, .8), (1, 1), (2, .4)\}$
$f(x) = x^2$
Then by applying the extension principle, we obtain

$$\tilde{B} = f(\tilde{A}) = \{(0, .8), (1, 1), (4, .4)\}$$

Figure 5–1 illustrates the relationship.

The extension principle as stated in definition 5–1 can and has been modified by using the algebraic sum (definition 3–8) rather than sup, and the product rather than min [Dubois and Prade 1980a]. Since, however, it is generally used as defined in definition 5–1, we will restrict our considerations to this "classical" version.

5.2 Operations for Type 2 Fuzzy Sets

The extension principle can be used to define set-theoretic operations for type 2 fuzzy sets as defined in definition 3–1.

We shall consider only fuzzy sets of type 2 with discrete domains. Let two fuzzy sets of type 2 be defined by

$$\tilde{A}(x) = \{(x, \mu_{\tilde{A}}(x))\} \quad \text{and} \quad \tilde{B}(x) = \{(x, \mu_{\tilde{B}}(x))\}$$

where

$$\mu_{\tilde{A}}(x) = \{(u_i, \mu_{ui}(x)) | x \in X, u_i, \mu_{ui}(x) \in [0, 1]\}$$
$$\mu_{\tilde{B}}(x) = \{(v_j, \mu_{vj}(x)) | x \in X, v_j, \mu_{vj}(x) \in [0, 1]\}$$

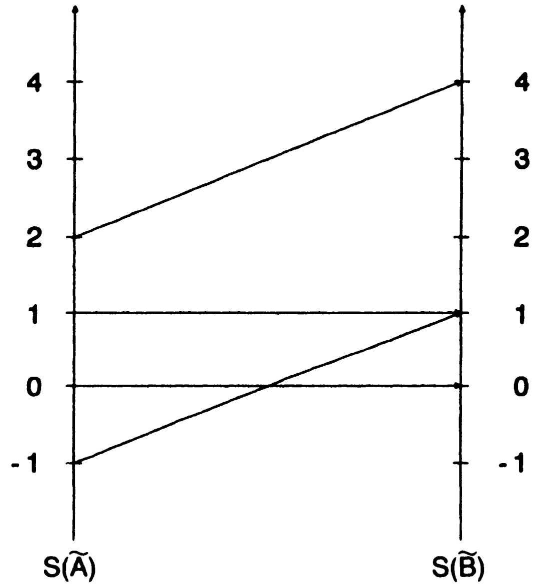


Figure 5–1. The extension principle.

The u_i and v_j are degrees of membership of type 1 fuzzy sets and the $\mu_{ui}(x)$ and $\mu_{vj}(x)$, respectively, their membership functions. Using the extension principle, the set-theoretic operations can be defined as follows [Mizumoto and Tanaka 1976]:

Definition 5–2

Let two *fuzzy sets of type 2* be defined as above. The membership function of their *union* is then defined by

$$\begin{aligned}\mu_{\tilde{A}\cup\tilde{B}}(x) &= \mu_{\tilde{A}}(x) \cup \mu_{\tilde{B}}(x) \\ &= \{(w, \mu_{\tilde{A}\cup\tilde{B}}(w)) | w = \max\{u_i, v_j\}, u_i, v_j \in [0,1]\}\end{aligned}$$

where

$$\mu_{\tilde{A}\cup\tilde{B}}(w) = \sup_{w=\max\{u_i, v_j\}} \min\{\mu_{ui}(x), \mu_{vj}(x)\}$$

Their *intersection* is defined by

$$\begin{aligned}\mu_{\tilde{A}\cap\tilde{B}}(x) &= \mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x) \\ &= \{(w, \mu_{\tilde{A}\cap\tilde{B}}(w)) | w = \min\{u_i, v_j\}, u_i, v_j \in [0,1]\}\end{aligned}$$

where

$$\mu_{\tilde{A}\cap\tilde{B}}(w) = \sup_{w=\min\{u_i,v_j\}} \min\{\mu_{u_i}(x), \mu_{vj}(x)\}$$

and the *complement* of $\tilde{A}$ by

$$\tilde{\mu}_{\complement\tilde{A}}(x) = \{[(1-u_i), \mu_{\tilde{A}}(u_i)]\}$$

Example 5–2

Let $X = 1, \ldots, 10$, $\tilde{A}$ = small integers
$\tilde{B}$ = integers close to 4

defined by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$$
$$\tilde{B} = \{(x, \mu_{\tilde{B}}(x))\}$$

where, for $x = 3$,

$$\mu_{\tilde{A}}(3) = \{(u_i, \mu_{ui}(3)) | i = 1, \ldots, 3\}$$
$$= \{(.8, 1), (.7, .5), (.6, .4)\}$$
$$\mu_{\tilde{B}}(3) = \{(v_j, \mu_{vj}(3)) | j = 1, \ldots, 3\}$$
$$= \{(1, 1), (.8, .5), (.7, .3)\}$$

Compute $\mu_{\tilde{A}\cap\tilde{B}}$:

u_i	v_j	$w = \min\{u_i, v_j\}$	$\mu_{ui}(3)$	$\mu_{vj}(3)$	$\min\{\mu_{ui}(3), \mu_{vj}(3)\}$
.8	1	.8	1	1	1
.8	.8	.8	1	.5	.5
.8	.7	.7	1	.3	.3
.7	1	.7	.5	1	.5
.7	.8	.7	.5	.5	.5
.7	.7	.7	.5	.3	.3
.6	1	.6	.4	1	.4
.6	.8	.6	.4	.5	.4
.6	.7	.6	.4	.3	.3

Next, compute the supremum of the degrees of membership of all pairs (u_i, v_j) that yield w as minimum:

$$\sup_{.8=\min\{u_i,v_j\}} \{1, .5\} = 1$$

$$\sup_{.7=\min\{u_i, v_j\}} \{.3, .5, .5, .3\} = .5$$

$$\sup_{.6=\min\{u_i, v_j\}} \{.4, .4, .3\} = .4$$

So we obtain the membership function of $x = 3$ as the fuzzy set

$$\mu_{\tilde{A}\cap\tilde{B}}(3) = \{(.8, 1), (.7, .5), (.6, 4)\}$$

Mizumoto and Tanaka [1976, p. 318] show that type 2 fuzzy sets as defined above are idempotent, commutative, and associative and satisfy the DeMorgan laws. They are, however, not distributive and do not satisfy the absorbtion laws, the identity laws, or the complement laws.

Example 5–2 is a good indication of the computational effort involved in operations with type 2 fuzzy sets. The reader should realize that in this example the degrees of membership of only *one* element of the type 2 fuzzy set is computed. For all other elements, such as $x = 4$, $x = 5$, . . . etc. of the sets $\tilde{A} * \tilde{B}$, the corresponding calculations would be necessary. Here "*" can be any set-theoretic operation mentioned so far.

5.3 Algebraic Operations with Fuzzy Numbers

Definition 5–3

A *fuzzy number* $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line $\mathbb{R}$ such that

1. It exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Nowadays, definition 5–3 is very often modified. For the sake of computational efficiency and ease of data acquisition, trapezoidal membership functions are often used. Figure 5–2 shows such a fuzzy set, which could be called "approximately 5" and which would normally be defined as the quadrupel {3, 4, 6, 7}. Strictly speaking, it is a fuzzy interval (see section 5.3.2). A triangular fuzzy number is, of course, a special case of this.

Definition 5–4

A fuzzy number $\tilde{M}$ is called *positive* (negative) if its membership function is such that $\mu_{\tilde{M}}(x) = 0$, $\forall x < 0$ ($\forall x > 0$).

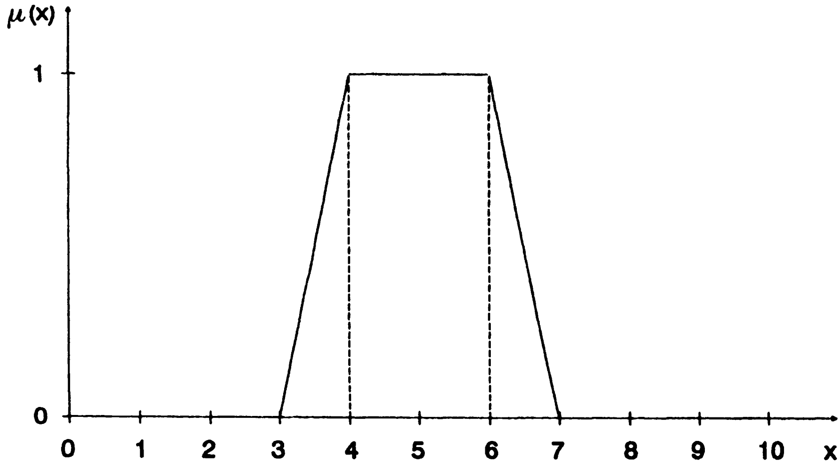


Figure 5–2. Trapezoidal "fuzzy number."

Example 5–3

The following fuzzy sets are fuzzy numbers:

$$\text{approximately } 5 = \{(3, .2), (4, .6), (5, 1), (6, .7), (7, .1)\}$$

$$\text{approximately } 10 = \{(8, .3), (9, .7), (10, 1), (11, .7), (12, .3)\}$$

But $\{(3, .8), (4, 1), (5, 1), (6, .7)\}$ is not a fuzzy number because $\mu(4)$ and also $\mu(5) = 1$.

We are all familiar with algebraic operations with crisp numbers. If we want to use fuzzy sets in applications, we will have to deal with fuzzy numbers, and the extension principle is one way to extend algebraic operations from crisp to fuzzy numbers.

We need a few more definitions: Let $F(\mathbb{R})$ be the set of real fuzzy numbers and $X = X_1 \times X_2$. We can define the following properties of binary operations:

Definition 5–5

A binary operation $*$ in $\mathbb{R}$ is called *increasing* (decreasing) if

$$\text{for} \quad x_1 > y_1 \quad \text{and} \quad x_2 > y_2$$
$$x_1 * x_2 > y_1 * y_2 \; (x_1 * x_2 < y_1 * y_2)$$

Example 5–4

$f(x, y) = x + y$ is an increasing operation.

$f(x, y) = x \cdot y$ is an increasing operation on $\mathbb{R}^+$.

$f(x, y) = -(x + y)$ is a decreasing operation.

If the normal algebraic operations $+, -, \cdot, :$ are extended to operations on fuzzy numbers, they shall be denoted by $\oplus, \ominus, \odot, \oslash$.

Theorem 5–1 [See Dubois and Prade 1980a, p. 44]

If $\tilde{M}$ and $\tilde{N}$ are fuzzy numbers whose membership functions are continuous and surjective from $\mathbb{R}$ to $[0, 1]$ and $*$ is a continuous increasing (decreasing) binary operation, then $\tilde{M} \circledast \tilde{N}$ is a fuzzy number whose membership function is continuous and surjective from $\mathbb{R}$ to $[0, 1]$.

Dubois and Prade [1980a] present procedures to determine the membership functions $\mu_{\tilde{M} \circledast \tilde{N}}$ on the basis of $\mu_{\tilde{M}}$ and $\mu_{\tilde{N}}$.

Theorem 5–2

If $\tilde{M}, \tilde{N} \in F(\mathbb{R})$ with $\mu_{\tilde{N}}(x)$ and $\mu_{\tilde{M}}(x)$ continuous membership functions, then by application of the extension principle for the binary operation $*$: $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$, the membership function of the fuzzy number $\tilde{M} \circledast \tilde{N}$ is given by

$$\mu_{\tilde{M} \circledast \tilde{N}}(z) = \sup_{z = x * y} \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}$$

Properties of the extended operation $\circledast$

Remark 5–1 [Dubois and Prade 1980a, p. 45]

1. For any commutative operation $*$, the extended operation $\circledast$ is also commutative.
2. For any associative operation $*$, the extended operation $\circledast$ is also associative.

5.3.1 Special Extended Operations

For unary operations f: $X \to Y$, $X = X_1$ (see definitions 5–1), the extension principle reduces for all $\tilde{M} \in F(\mathbb{R})$ to

$$\mu_{f(\tilde{M})}(z) = \sup_{x \in f^{-1}(z)} \mu_{\tilde{M}}(x)$$

Example 5–5

1. For $f(x) = -x$, the opposite of a fuzzy number $\tilde{M}$ is given by $-\tilde{M} = \{(x, \mu_{-\tilde{M}}(x))|x \in X\}$, where $\mu_{-\tilde{M}}(x) = \mu_{\tilde{M}}(-x)$.
2. If $f(x) = \frac{1}{x}$, then the inverse of a fuzzy number $\tilde{M}$ is given by $\tilde{M}^{-1} = \{(x, \mu_{\tilde{M}}^{-1}(x))|x \in X\}$, where $\mu_{\tilde{M}}^{-1}(x) = \mu_{\tilde{M}}(\frac{1}{x})$.
3. For $\lambda \in \mathbb{R}\backslash\{0\}$ and $f(x) = \lambda \cdot x$, then the scalar multiplication of a fuzzy number is given by $\lambda\tilde{M} = \{(x, \mu_{\lambda\tilde{M}}(x))|x \in X\}$, where $\mu_{\lambda\tilde{M}}(x) = \mu_{\tilde{M}}(\lambda \cdot x)$.

In the following, we shall apply the extension principle to binary operations. A generalization to n-ary operations is straightforward.

Extended Addition. Since addition is an increasing operation according to theorem 5–1, we get for the extended addition $\oplus$ of fuzzy numbers that $f(\tilde{N}, \tilde{M}) = \tilde{N} \oplus \tilde{M}$, $\tilde{N}, \tilde{M} \in F(\mathbb{R})$ is a fuzzy number—that is, $\tilde{N} \oplus \tilde{M} \in F(\mathbb{R})$.

Properties of $\oplus$

1. $\ominus(\tilde{M} \oplus \tilde{N}) = (\ominus\tilde{M}) \oplus (\ominus\tilde{N})$.
2. $\oplus$ is commutative.
3. $\oplus$ is associative.
4. $0 \in \mathbb{R} \subseteq F(\mathbb{R})$ is the neutral element for $\oplus$, that is, $\tilde{M} \oplus 0 = \tilde{M}$, $\forall\tilde{M} \in F(\mathbb{R})$.
5. For $\oplus$ there does not exist an inverse element, that is, $\forall\tilde{M} \in F(\mathbb{R})\backslash\mathbb{R}$: $\tilde{M} \oplus (\ominus\tilde{M}) \neq 0 \in \mathbb{R}$.

One of the consequences [Yager 1980] is that fuzzy equations are very difficult to solve because the variables cannot be eliminated as usual.

Extended Product. Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, according to theorem 5–1, the product of positive fuzzy numbers or of negative fuzzy numbers results in a positive fuzzy number. Let $\tilde{M}$ be a positive and $\tilde{N}$ a negative fuzzy number. Then $\ominus\tilde{M}$ is also negative and $\tilde{M} \odot \tilde{N} = \ominus(\ominus\tilde{M} \odot \tilde{N})$ results in a negative fuzzy number.

Properties of $\odot$

1. $(\ominus\tilde{M}) \odot \tilde{N} = \ominus(\tilde{M} \odot \tilde{N})$.
2. $\odot$ is commutative.
3. $\odot$ is associative.

4. $\tilde{M} \odot 1 = \tilde{M}$, $1 \in \mathbb{R} \subseteq F(\mathbb{R})$ is the neutral element for $\odot$, that is, $\tilde{M} \odot 1 = \tilde{M}$, $\forall \tilde{M} \in F(\mathbb{R})$.
5. For $\odot$ there does not exist an inverse element, that is, $\forall \tilde{M} \in F(\mathbb{R}) \backslash \mathbb{R}$: $\tilde{M} \odot \tilde{M}^{-1} \neq 1$.

Theorem 5–3 [for the proof, see Dubois and Prade 1980a, p. 51]

If $\tilde{M}$ is either a positive or a negative fuzzy number and $\tilde{N}$ and $\tilde{P}$ are both either positive or negative fuzzy numbers, then

$$\tilde{M} \odot (\tilde{N} \oplus \tilde{P}) = (\tilde{M} \odot \tilde{N}) \oplus (\tilde{M} \odot \tilde{P})$$

Extended Subtraction. Subtraction is neither an increasing nor a decreasing operation. Therefore theorem 5–1 is not immediately applicable. The operation $\tilde{M} \ominus \tilde{N}$ can, however, always be written as $\tilde{M} \ominus \tilde{N} = \tilde{M} \oplus (\ominus \tilde{N})$.

Applying the extension principle [Dubois and Prade 1979] yields

$$\begin{aligned}
\mu_{\tilde{M} \ominus \tilde{N}}(z) &= \sup_{z=x-y} \min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)) \\
&= \sup_{z=x+y} \min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(-y)) \\
&= \sup_{z=x+y} \min(\mu_{\tilde{M}}(x), \mu_{-\tilde{N}}(y))
\end{aligned}$$

Thus $\tilde{M} \ominus \tilde{N}$ is a fuzzy number whenever $\tilde{M}$ and $\tilde{N}$ are.

Extended Division. Division is also neither an increasing nor a decreasing operation. If $\tilde{M}$ and $\tilde{N}$ are strictly positive fuzzy numbers, however (that is, $\mu_{\tilde{M}}(x) = 0$ and $\mu_{\tilde{N}}(x) = 0$ $\forall x \leq 0$), we obtain in analogy to the extended subtraction

$$\begin{aligned}
\mu_{\tilde{M} \oslash \tilde{N}}(z) &= \sup_{z=x/y} \min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)) \\
&= \sup_{z=xy} \min\left(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}\left(\frac{1}{y}\right)\right) \\
&= \sup_{z=xy} \min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}^{-1}(y))
\end{aligned}$$

$\tilde{N}^{-1}$ is a positive fuzzy number. Hence theorem 5–1 can now be applied. The same is true if $\tilde{M}$ and $\tilde{N}$ are both strictly negative fuzzy numbers.

Similar results can be obtained by using other than the min-max operations—for instance, those of definitions 3–7 through 3–11.

Extended operations with fuzzy numbers involve rather extensive computations as long as no restrictions are put on the type of membership functions allowed. Dubois and Prade [1979] propose a general algorithm for performing extended operations. For practical purposes, however, it will generally be more appropriate to resort to specific kinds of fuzzy numbers, as they are described in the next section. The generality is not limited considerably by limiting extended operations to fuzzy numbers in *LR*-representation or even to triangular fuzzy numbers [van Laarhoven and Pedrycz 1983], and the computational effort is very much decreased. The reader should also realize that extended operations on the basis of min-max cannot be directly applied to "fuzzy numbers" with discrete supports. As illustrated by example 5–6, the resulting fuzzy sets may no longer be convex and therefore no longer considered as fuzzy numbers.

Example 5–6

Let $\tilde{M} = \{(1, .3), (2, 1), (3, .4)\}$
$\tilde{N} = \{(2, .7), (3, 1), (4, .2)\}$
Then

$\tilde{M} \odot \tilde{N} = \{(2, .3), (3, .3), (4, .7), (6, 1), (8, .2), (\underline{9, .4}), (12, .2)\}$

5.3.2 Extended Operations for LR-Representation of Fuzzy Sets

Computational efficiency is of particular importance when using fuzzy set theory to solve real problems, that is, problems of realistic size. In the following, therefore, we shall consider in detail the *LR*-representation of fuzzy sets, which increases computational efficiency without limiting the generality beyond acceptable limits.

Dubois and Prade [1979] suggest a special type of representation for fuzzy numbers of the following type: They call L (and R), which map $\mathbb{R}^+ \to [0, 1]$, and are decreasing, *shape functions* if $L(0) = 1$, $L(x) < 1$ for $\forall x > 0$; $L(x) > 0$ for $\forall x < 1$; $L(1) = 0$ or $[L(x) > 0, \forall x$ and $L(+\infty) = 0]$.

Definition 5–6

A *fuzzy number* $\tilde{M}$ is of *LR-type* if there exist reference functions L (for left), R (for right), and scalars $\alpha > 0$, $\beta > 0$ with

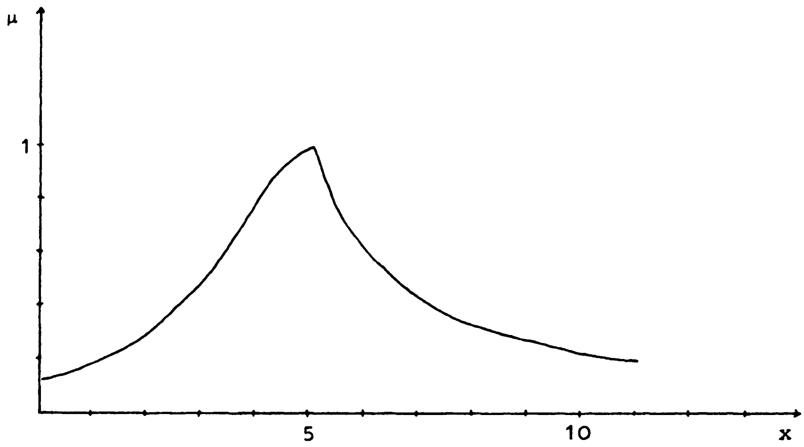


Figure 5–3. *LR*-representation of fuzzy numbers.

$$\mu_{\tilde{M}}(x)\begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & \text{for} \quad x \le m \\ R\left(\dfrac{x-m}{\beta}\right) & \text{for} \quad x \ge m \end{cases}$$

m, called the mean value of $\tilde{M}$, is a real number, and α and β are called the left and right spreads, respectively. Symbolically, $\tilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$. (See figure 5–3.)

For $L(z)$, different functions can be chosen. Dubois and Prade [1988a, p. 50] mention, for instance, $L(x) = \max(0, 1 - x)^p$, $L(x) = \max(0, 1 - x^p)$, with $p > 0$ and $L(x) = e^{-x}$ or $L(x) = e^{-x^2}$. These examples already give an impression of the wide scope of $L(z)$. One problem, of course, is to find the appropriate function in a specific context.

Example 5–7

Let

$$L(x) = \frac{1}{1+x^2}$$

$$R(x) = \frac{1}{1+2|x|}$$

$$\alpha = 2, \beta = 3, m = 5$$

Then

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\dfrac{5-x}{2}\right) = \dfrac{1}{1+\left(\dfrac{5-x}{2}\right)^2} & \text{for} \quad x \le 5 \\ R\left(\dfrac{x-5}{3}\right) = \dfrac{1}{1+\left|\dfrac{2(x-5)}{3}\right|} & \text{for} \quad x \ge 5 \end{cases}$$

If the m is not a real number but an interval $[\underline{m}, \overline{m}]$, then the fuzzy set $\tilde{M}$ is not a fuzzy number but a fuzzy interval. Accordingly, a fuzzy interval in *LR*-representation can be defined as follows:

Definition 5–6a [Dubois and Prade 1988a, p. 48]

A *fuzzy interval* $\tilde{M}$ is of *LR*-type if there exist shape functions L and R and four parameters $(\underline{m}, \overline{m}) \in \mathbb{R}^2 \cup \{-\infty, +\infty\}$, α, β and the membership function of $\tilde{M}$ is

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\dfrac{\underline{m}-x}{\alpha}\right) & \text{for} \quad x \le \underline{m} \\ 1 & \text{for} \quad \underline{m} \le x \le \overline{m} \\ R\left(\dfrac{x-\overline{m}}{\beta}\right) & \text{for} \quad x \ge \underline{m} \end{cases}$$

The fuzzy interval is then denoted by

$$\tilde{M} = (\underline{m}, \overline{m}, \alpha, \beta)_{LR}$$

This definition is very general and allows quantification of quite different types of information; for instance, if $\tilde{M}$ is supposed to be a real crisp number for $m \in \mathbb{R}$,

$$\tilde{M} = (m, m, 0, 0)_{LR}, \forall L, \forall R$$

If $\tilde{M}$ is a crisp interval,

$$\tilde{M} = (a, b, 0, 0)_{LR}, \forall L, \forall R$$

and if $\tilde{M}$ is a "trapezoidal fuzzy number" (see definition 5–3), $L(x) = R(x) = \max(0, 1 - x)$ is implied.

For *LR* fuzzy numbers, the computations necessary for the above-mentioned operations are considerably simplified: Dubois and Prade [1979] showed that exact formulas can be given for $\oplus$ and $\ominus$. They also suggested approximate expressions for $\odot$ and $\oslash$ [Dubois and Prade 1979], which approximate better when the spreads are smaller compared to the mean values.

Theorem 5–4

Let $\tilde{M}$, $\tilde{N}$ be two fuzzy numbers of *LR*-type:

$$\tilde{M} = (m, \alpha, \beta)_{LR}, \quad \tilde{N} = (n, \gamma, \delta)_{LR}$$

Then

1. $(m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$.
2. $-(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{LR}$.
3. $(m, \alpha, \beta)_{LR} \ominus (n, \gamma, \delta)_{LR} = (m - n, \alpha + \delta, \beta + \gamma)_{LR}$.

Example 5–8

$$\begin{aligned} L(x) = R(x) &= \frac{1}{1+x^2} \\ \tilde{M} &= (1, .5, .8)_{LR} \\ \tilde{N} &= (2, .6, .2)_{LR} \\ \tilde{M} \oplus \tilde{N} &= (3, 1.1, 1)_{LR} \\ \tilde{O} &= (2, .6, .2)_{LR} \\ \ominus \tilde{O} &= (-2, .2, .6)_{LR} \\ \tilde{M} \ominus \tilde{O} &= (-1, .7, 1.4)_{LR} \end{aligned}$$

Theorem 5–5 [Dubois and Prade 1980a, p. 55]

Let $\tilde{M}$, $\tilde{N}$ be fuzzy numbers as in definition 5–3; then

$$(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}$$

for $\tilde{M}$, $\tilde{N}$ positive;

$$(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR}$$

for $\tilde{N}$ positive, $\tilde{M}$ negative, and

$$(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, -n\beta - m\delta, n\alpha - m\gamma)_{LR}$$

for $\tilde{M}$, $\tilde{N}$ negative.

The following example shows an application of theorem 5–5.

Example 5–9

Let $\tilde{M} = (2, .2, .1)_{LR}$
$\tilde{N} = (3, .1, .3)_{LR}$
be fuzzy numbers of *LR*-type with reference functions

$$L(z) = R(z) = \begin{cases} 1 & -1 \le z \le 1 \\ 0 & \text{else} \end{cases}$$

If we are interested in the LR-representation of $\tilde{M} \odot \tilde{N}$, we prove the conditions of theorem 5–5 and apply it. Thus, with

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\frac{2-x}{.2}\right) & x \le 2 \\ R\left(\frac{x-2}{.1}\right) & x \ge 2 \end{cases}$$

$$= \begin{cases} 1 & -1 \le \frac{2-x}{.2} \le 1 \quad \text{and} \quad -1 \le \frac{x-2}{.1} \le 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 & 1.9 \le x \le 2.1 \\ 0 & \text{else} \end{cases}$$

it follows that $\tilde{M}$ is positive.

$$\mu_{\tilde{N}}(x) = \begin{cases} L\left(\frac{3-x}{.1}\right) & x \le 3 \\ R\left(\frac{x-3}{.3}\right) & x \ge 3 \end{cases}$$

$$= \begin{cases} 1 & 2.9 \le x \le 3.1 \\ 0 & \text{else} \end{cases}$$

shows that $\tilde{N}$ is positive.

Following theorem 5–5 for the case in which $\tilde{M}$ and $\tilde{N}$ are positive, we obtain

$$\tilde{M} \odot \tilde{N} \approx (2 \cdot 3, 2 \cdot .1 + 3 \cdot .2, 2 \cdot .3 + 3 \cdot .1)_{LR} = (6, .8, .9)_{LR}$$

Exercises

1. Let $X = \mathbb{N} \times \mathbb{N}$
   $\tilde{A}_1 = \{(1, .6), (2, .8), (3, 1), (4, .6)\}$
   $\tilde{A}_2 = \{(0, .5), (1, .7), (2, .9), (3, 1), (4, .4)\}$
   $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by

   $$f(x, y) = z, x \in \tilde{A}_1, y \in \tilde{A}_2$$

   Determine the image $f(\tilde{A}_1 \times \tilde{A}_2)$ by the extension principle.
2. Compute $\mu_{\tilde{A} \cup \tilde{B}}$ and $\mu_{\complement \tilde{A}}$ for $\tilde{A}$, $\tilde{B}$ as in example 5–2.

3. Which of the following fuzzy sets are fuzzy numbers?

   a. $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in \mathbb{R}\}$

   where

$$\mu_{\tilde{A}}(x) = \begin{cases} \left(\left(1+\dfrac{5-x}{2}\right)^2\right)^{-1} & x \le 5 \\ \left(1+\left|\dfrac{2(x-5)}{3}\right|\right)^{-1} & x \ge 5 \end{cases}$$

   b. $\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) | x \in \mathbb{R}^+\}$

   where

$$\mu_{\tilde{B}}(x) = \begin{cases} x & x \in [0, 1] \\ 1 & x \in [1, 2] \\ 3-x & x \in [2, 3] \end{cases}$$

   c. $\tilde{C} = \{(0, .4), (1, 1), (2, .7)\}$

4. Which of the following functions are reference functions for $x \in \mathbb{R}$?

   a. $f_1(x) = |x + 1|$

   b. $f_2(x) = \dfrac{1}{1+x^2}$

   c. $f_3(x) = \begin{cases} \dfrac{1}{2}x+1 & x \in [-2, 0] \\ -2x+1 & x \in \left[0, \dfrac{1}{2}\right] \\ 0 & \text{else} \end{cases}$

   d. $f_4(x) = \dfrac{1}{1+a\,|x|^p}\quad p \ge 1$

5. Let $\tilde{M}$, $L(x)$, $R(x)$ be defined as in example 5–8. $\tilde{N} = (-4, .1, .6)_{LR}$. Compute $\tilde{M} \ominus \tilde{N}$.
6. Let $\tilde{M}$, $\tilde{N}$ be defined as in example 5–8. Compute $\tilde{M} \odot \tilde{N}$.
7. Develop an approximate formula to compute $\tilde{M} \odot \tilde{N}$, $\tilde{M} = (m, \alpha, \beta)_{LR}$, $\tilde{N} = (n, \gamma, \sigma)_{LR}$. (Remember how the formula was derived for the general extended division.)

6 FUZZY RELATIONS AND FUZZY GRAPHS

6.1 Fuzzy Relations on Sets and Fuzzy Sets

Fuzzy relations are fuzzy subsets of $X \times Y$, that is, mappings from $X \to Y$. They have been studied by a number of authors, in particular by Zadeh [1965, 1971], Kaufmann [1975], and Rosenfeld [1975]. Applications of fuzzy relations are widespread and important. We shall consider some of them and point to more possible uses at the end of this chapter. We shall exemplarily consider only binary relations. A generalization to n-ary relations is straightforward.

Definition 6–1

Let $X, Y \subseteq \mathbb{R}$ be universal sets; then

$$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) | (x, y) \in X \times Y\}$$

is called a *fuzzy relation* on $X \times Y$.

Example 6–1

Let $X = Y = \mathbb{R}$ and $\tilde{R}$: = "considerably larger than." The membership function of the fuzzy relation, which is, of course, a fuzzy set on $X \times Y$, can then be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ \dfrac{(x-y)}{10y} & \text{for } y < x \leq 11y \\ 1 & \text{for } x > 11y \end{cases}$$

A different membership function for this relation could be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ \left(1 + (y-x)^{-2}\right)^{-1} & \text{for } x > y \end{cases}$$

For discrete supports, fuzzy relations can also be defined by matrixes.

Example 6–2

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$

$\tilde{R}$ = "x considerably larger than y":

	y_1	y_2	y_3	y_4
x_1	.8	1	.1	.7
x_2	0	.8	0	0
x_3	.9	1	.7	.8

and

$\tilde{Z}$ = "y very close to x":

	y_1	y_2	y_3	y_4
x_1	.4	0	.9	.6
x_2	.9	.4	.5	.7
x_3	.3	0	.8	.5

In definition 6–1 it was assumed that $\mu_{\tilde{R}}$ was a mapping from $X \times Y$ to [0, 1]; that is, the definition assigns to each pair (x, y) a degree of membership in the unit interval. In some instances, such as in graph theory, it is useful to consider fuzzy relations that map from fuzzy sets contained in the universal sets into the unit interval. Then definition 6–1 has to be generalized [Rosenfeld 1975].

Definition 6–2

Let $X, Y \subseteq \mathbb{R}$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\},$$
$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y \in Y\}, \quad \text{two fuzzy sets.}$$

Then $\tilde{R} = \{[(x, y), \mu_{\tilde{R}}(x, y)] | (x, y) \in X \times Y\}$ is a *fuzzy relation* on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{A}}(x), \forall (x, y) \in X \times Y$$

and

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{B}}(y), \forall (x, y) \in X \times Y.$$

This definition will be particularly useful when defining fuzzy graphs: Let the elements of the fuzzy relation of definition 6–2 be the nodes of a fuzzy graph that is represented by this fuzzy relation. The degrees of membership of the elements of the related fuzzy sets define the "strength" of or the flow in the respective nodes of the graph, while the degrees of membership of the corresponding pairs in the relation are the "flows" or "capacities" of the edges. The additional requirement of definition 6–2 ($\mu_{\tilde{R}}(x, y) \leq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$) then ensures that the "flows" in the edges of the graph can never exceed the flows in the respective nodes.

Fuzzy relations are obviously fuzzy sets in product spaces. Therefore set-theoretic and algebraic operations can be defined for them in analogy to the definitions in chapters 2 and 3 by utilizing the extension principle.

Definition 6–3

Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product space. The *union/intersection* of $\tilde{R}$ with $\tilde{Z}$ is then defined by

$$\mu_{\tilde{R} \cup \tilde{Z}}(x, y) = \max\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, (x, y) \in X \times Y$$
$$\mu_{\tilde{R} \cap \tilde{Z}}(x, y) = \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, (x, y) \in X \times Y$$

Example 6–3

Let $\tilde{R}$ and $\tilde{Z}$ be the two fuzzy relations defined in example 6–2. The union of $\tilde{R}$ and $\tilde{Z}$, which can be interpreted as "x considerably larger or very close to y," is then given by

$\tilde{R} \cup \tilde{Z}$:

	y_1	y_2	y_3	y_4
x_1	.8	1	.9	.7
x_2	.9	.8	.5	.7
x_3	.9	1	.8	.8

The intersection of $\tilde{R}$ and $\tilde{Z}$ is represented by

$\tilde{R} \cap \tilde{Z}$:

	y_1	y_2	y_3	y_4
x_1	.4	0	.1	.6
x_2	0	.4	0	0
x_3	.3	0	.7	.5

So far, "min" and "max" have been used to define intersection and union. Since fuzzy relations are fuzzy sets, operations can also be defined using the alternative definitions in section 3.2. Some additional concepts, such as the projection and the cylindrical extension of fuzzy relations, have been shown to be useful.

Definition 6–4

Let $\tilde{R} = \{[(x, y), \mu_{\tilde{R}}(x, y)] \mid (x, y) \in X \times Y\}$ be a fuzzy binary relation. The *first projection* of $\tilde{R}$ is then defined as

$$\tilde{R}^{(1)} = \{(x, \max_y \mu_{\tilde{R}}(x, y)) | (x, y) \in X \times Y\}$$

The *second projection* is defined as

$$\tilde{R}^{(2)} = \{(y, \max_x \mu_{\tilde{R}}(x, y)) | (x, y) \in X \times Y\}$$

and the *total projection* as

$$\tilde{R}^{(T)} = \max_x \max_x \{\mu_{\tilde{R}}(x, y) | (x, y) \in X \times Y\}$$

Example 6–4

Let $\tilde{R}$ be a fuzzy relation defined by the following relational matrix. The first, second, and total projections are then shown at the appropriate places below.

	y_1	y_2	y_3	y_4	y_5	y_6	First projection $[\mu_{\tilde{R}^{(1)}}(x)]$
x_1	.1	.2	.4	.8	1	.8	1
$\tilde{R}$: x_2	.2	.4	.8	1	.8	.6	1
x_3	.4	.8	1	.8	.4	.2	1
Second projection: $[\mu_{\tilde{R}^{(2)}}(x)]$	.4	.8	1	1	1	.8	1 Total projection

The relation resulting from applying an operation of projection to another relation is also called a "shadow" [Zadeh 1973a]. Let us now consider a more general space, namely, $X = X_1 \times \ldots \times X_n$; and let $\tilde{R}_q$ be a projection on $X_{i_1} \times \ldots \times X_{i_k}$, where $(i_1, \ldots, i_k)$ is a subsequence of $(1, \ldots, n)$. It is obvious that distinct fuzzy relations in the same universe can have the same projection. There must, however, be a uniquely defined largest relation $\tilde{R}_{qL}$ $(X_1, \ldots, X_n)$ with $\mu_{\tilde{R}_{qL}}$ $(X_{i_1}, \ldots, X_{i_k})$ for each projection. This largest relation is called the *cylindrical extension of the projection relation.*

Definition 6–5

$\tilde{R}_{qL} \subseteq X$ is the largest relation in X of which the projection is $\tilde{R}_q$, $\tilde{R}_{qL}$ is then called the *cylindrical extension* of $\tilde{R}_q$ and $\tilde{R}_q$ is the base of $\tilde{R}_{qL}$.

Example 6–5

The cylindrical extension of $R^{(2)}$ (example 6–4) is

	y_1	y_2	y_3	y_4	y_5	y_6
x_1	.4	.8	1	1	1	.8
$\tilde{R}_2$: x_2	.4	.8	1	1	1	.8
x_3	.4	.8	1	1	1	.8

Definition 6–6

Let $\tilde{R}$ be a fuzzy relation on $X = X_1 \times \ldots \times X_n$ and $\tilde{R}_1$ and $\tilde{R}_2$ be two fuzzy projections on $X_1 \times \ldots \times X_r$ and $X_s \times \ldots \times X_n$, respectively, with $s \leq r + 1$ and $\tilde{R}_{1L}$, $\tilde{R}_{2L}$ their respective cylindrical extensions.

The meet of $\tilde{R}_1$ and $\tilde{R}_2$ is then defined as $\tilde{R}_{1L} \cap \tilde{R}_{2L}$ and their join as $\tilde{R}_{1L} \cup \tilde{R}_{2L}$.

6.1.1 Compositions of Fuzzy Relations

Fuzzy relations in different product spaces can be combined with each other by the operation "composition." Different versions of "composition" have been suggested, which differ in their results and also with respect to their mathematical properties. The max-min composition has become the best known and the most frequently used one. However, often the so-called max-product or max-average compositions lead to results that are more appealing.

Definition 6–7

Max-min composition: Let $\tilde{R}_1(x, y)$, $(x, y) \in X \times Y$ and $\tilde{R}_2(y, z)$, $(y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition $\tilde{R}_1$ max-min $\tilde{R}_2$ is then the fuzzy set

$$\tilde{R}_1 \circ \tilde{R}_2 = \{[(x, z), \max_y\{\min\{\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z)\}\}] | x \in X, y \in Y, z \in Z\}$$

$\mu_{\tilde{R}_1 \circ \tilde{R}_2}$ is again the membership function of a fuzzy relation on fuzzy sets (definition 6–2).

A more general definition of composition is the "max-* composition."

Definition 6–8

Let $\tilde{R}_1$ and $\tilde{R}_2$ be defined as in definition 6–7. The *max-* composition* of $\tilde{R}_1$ and $\tilde{R}_2$ is then defined as

$$\tilde{R}_1 \overset{\circ}{*} \tilde{R}_2 = \{[(x, z), \max_y(\mu_{\tilde{R}_1}(x, y) * \mu_{\tilde{R}_2}(y, z))] | x \in X, y \in Y, z \in Z\}$$

If * is an associative operation that is monotonically nondecreasing in each argument, then the max-* composition corresponds essentially to the max-min composition. Two special cases of the max-* composition are proposed in the next definition.

Definition 6–9

[Rosenfeld 1975]: Let $\tilde{R}_1$ and $\tilde{R}_2$, respectively, be defined as in definition 6–7. The *max-prod composition* $\tilde{R}_1 \overset{\circ}{\cdot} \tilde{R}_2$ and the *max-av composition* $\tilde{R}_1 \underset{\text{av}}{\circ} \tilde{R}_2$ are then defined as follows:

$$\tilde{R}_1 \overset{\circ}{\cdot} \tilde{R}_2(x, z) = \max_y [\mu_{\tilde{R}_1}(x, y) \cdot \mu_{\tilde{R}_2}(y, z) | x \in X, y \in Y, z \in Z]$$

$$\tilde{R}_1 \underset{\text{av}}{\circ} \tilde{R}_2(x, z) = \tfrac{1}{2} \cdot \max_{Y} [\mu_{\tilde{R}_1}(x, y) \cdot \mu_{\tilde{R}_2}(y, z) | x \in X, y \in Y, z \in Z]$$

Example 6–6

Let $\tilde{R}_1(x, y)$ and $\tilde{R}_2(y, z)$ be defined by the following relational matrixes [Kaufmann 1975, p. 62]:

$\tilde{R}_1$:	y_1	y_2	y_3	y_4	y_5
x_1	.1	.2	0	1	.7
x_2	.3	.5	0	.2	1
x_3	.8	0	1	.4	.3

$\tilde{R}_2$:	z_1	z_2	z_3	z_4
y_1	.9	0	.3	.4
y_2	.2	1	.8	0
y_3	.8	0	.7	1
y_4	.4	.2	.3	0
y_5	0	1	0	.8

We shall first compute the min-max-composition $\tilde{R}_1 \circ \tilde{R}_2(x, z)$. We shall show in detail the determination for $x = x_1$, $z = z_1$ and leave it to the reader to verify the total results shown in the matrix at the end of the detailed computations. We first perform the min operation in the minor brackets of definition 6–7:

Let $x = x_1$, $z = z_1$, and $y = y_i$, $i = 1, \ldots, 5$:

$$\min\{\mu_{\tilde{R}_1}(x_1, y_1), \mu_{\tilde{R}_2}(y_1, z_1)\} = \min\{.1, .9\} = .1$$

$$\min\{\mu_{\tilde{R}_1}(x_1, y_2), \mu_{\tilde{R}_2}(y_2, z_1)\} = \min\{.2, .2\} = .2$$

$$\min\{\mu_{\tilde{R}_1}(x_1, y_3), \mu_{\tilde{R}_2}(y_3, z_1)\} = \min\{0, .8\} = 0$$

$$\min\{\mu_{\tilde{R}_1}(x_1, y_4), \mu_{\tilde{R}_2}(y_4, z_1)\} = \min\{1, .4\} = 4$$

$$\min\{\mu_{\tilde{R}_1}(x_1, y_5), \mu_{\tilde{R}_2}(y_5, z_1)\} = \min\{.7, 0\} = 0$$

$$\begin{aligned}\tilde{R}_1 \circ \tilde{R}_2(x_1, z_1) &= ((x_1, z_1), \mu_{\tilde{R}_1 \circ \tilde{R}_2}(x_1, z_1)) \\ &= ((x_1, z_1), \max\{.1, .2, 0, .4, 0\}) = ((x_1, z_1), .4)\end{aligned}$$

In analogy to the above computation we now determine the grades of membership for all pairs (x_i, z_j), $i = 1, \ldots, 3, j = 1, \ldots, 4$ and arrive at

$\tilde{R}_1 \circ \tilde{R}_2$:

	z_1	z_2	z_3	z_4
x_1	.4	.7	.3	.7
x_2	.3	1	.5	.8
x_3	.8	.3	.7	1

For the max-prod, we obtain

$$x = x_1,\ z = z_1,\ y = y_i,\ i = 1, \ldots, 5:$$

$$\mu_{\tilde{R}_1}(x_1, y_1) \cdot \mu_{\tilde{R}_2}(y_1, z_1) = .1 \cdot .9 = .09$$

$$\mu_{\tilde{R}_1}(x_1, y_2) \cdot \mu_{\tilde{R}_2}(y_2, z_1) = .2 \cdot .2 = .04$$

$$\mu_{\tilde{R}_1}(x_1, y_3) \cdot \mu_{\tilde{R}_2}(y_3, z_1) = 0 \cdot .8 = 0$$

$$\mu_{\tilde{R}_1}(x_1, y_4) \cdot \mu_{\tilde{R}_2}(y_4, z_1) = 1 \cdot .4 = 4$$

$$\mu_{\tilde{R}_1}(x_1, y_5) \cdot \mu_{\tilde{R}_2}(y_5, z_1) = .7 \cdot 0 = 0$$

Hence

$$\begin{aligned} \tilde{R}_1 \overset{\circ}{\cdot} \tilde{R}_2(x_1, z_1) &= ((x_1, z_1), (\mu_{\tilde{R}_1 \circ \tilde{R}_2}(x_1, z_1))) \\ &= ((x_1, z_1), \max\{.09, .04, 0, .4, 0\}) \\ &= ((x_1, z_1), .4) \end{aligned}$$

After performing the remaining computations, we obtain

$\tilde{R}_1 \overset{\circ}{\cdot} \tilde{R}_2$:

	z_1	z_2	z_3	z_4
x_1	.4	.7	.3	.56
x_2	.27	1	.4	.8
x_3	.8	.3	.7	1

The max-av composition finally yields

i	$\mu(x_1, y_i) + \mu(y_i, z_1)$
1	1
2	.4
3	.8
4	1.4
5	.7

Hence

$$\tfrac{1}{2} \cdot \max_y \{\mu_{\tilde{R}_1}(x_1, y_i) + \mu_{\tilde{R}_2}(y_i, z_1)\} = \tfrac{1}{2} \cdot (1.4) = .7$$

$\tilde{R}_1 \underset{\text{av}}{\circ} \tilde{R}_2$:

	z_1	z_2	z_3	z_4
x_1	.7	.85	.65	.75
x_2	.6	1	.65	.9
x_3	.9	.65	.85	1

6.1.2 Properties of the Min-Max Composition

(For proofs and more details see, for instance, Rosenfeld 1975.)

Associativity. The max-min composition is *associative*, that is,

$$(\tilde{R}_3 \circ \tilde{R}_2) \circ \tilde{R}_1 = \tilde{R}_3 \circ (\tilde{R}_2 \circ \tilde{R}_1).$$

Hence $\tilde{R}_1 \circ \tilde{R}_1 \circ \tilde{R}_1 = \tilde{R}_1^3$, and the third power of a fuzzy relation is defined.

Reflexivity

Definition 6–10

Let $\tilde{R}$ be a fuzzy relation in $X \times X$.

1. $\tilde{R}$ is called *reflexive* [Zadeh 1971] if

$$\mu_{\tilde{R}}(x, x) = 1 \; \forall x \in X$$

2. $\tilde{R}$ is called *ε-reflective* [Yeh 1975] if

$$\mu_{\tilde{R}}(x, x) \geq \varepsilon \; \forall x \in X$$

3. $\tilde{R}$ is called *weakly reflexive* [Yeh 1975] if

$$\left.\begin{array}{l} \mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x) \\ \mu_{\tilde{R}}(y, x) \leq \mu_{\tilde{R}}(x, x) \end{array}\right\} \forall x, y \in X.$$

Example 6–7

Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.

The following relation "y is close to x" is reflexive:

$\tilde{R}$	y_1	y_2	y_3	y_4
x_1	1	0	.2	.3
x_2	0	1	.1	1
x_3	.2	.7	1	.4
x_4	0	1	.4	1

If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive fuzzy relations, then the max-min composition $\tilde{R}_1 \circ \tilde{R}_2$ is also reflexive.

Symmetry

Definition 6–11

A fuzzy relation $\tilde{R}$ is called *symmetric* if $\tilde{R}(x, y) = \tilde{R}(y, x)\ \forall x, y \in X$.

Definition 6–12

A relation is called *antisymmetric* if for

$$x \neq y \quad \left.\begin{array}{ll} \text{either} & \mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x) \\ \text{or} & \mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(x, x) = 0 \end{array}\right\} \forall x, y \in X$$

[Kaufmann 1975, p. 105].
A relation is called *perfectly antisymmetric* if for $x \neq y$ whenever

$$\mu_{\tilde{R}}(x, y) > 0 \quad \text{then} \quad \mu_{\tilde{R}}(y, x) = 0\ \forall x, y \in X$$

[Zadeh 1971].

Example 6–8

$\tilde{R}_1$	x_1	x_2	x_3	x_4
x_1	.4	0	.1	.8
x_2	.8	1	0	0
x_3	0	.6	.7	0
x_4	0	.2	0	0

$$\tilde{R}_2: \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & .4 & 0 & .7 & 0 \\ x_2 & 0 & 1 & .9 & .6 \\ x_3 & .8 & .4 & .7 & .4 \\ x_4 & 0 & .1 & 0 & 0 \end{array}$$

$$\tilde{R}_3: \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & .4 & .8 & .1 & .8 \\ x_2 & .8 & 1 & .0 & .2 \\ x_3 & .1 & .6 & .7 & .1 \\ x_4 & 0 & .2 & 0 & 0 \end{array}$$

$\tilde{R}_1$ is a perfectly antisymmetric relation, while $\tilde{R}_2$ is an antisymmetric, but not perfectly antisymmetric relation. $\tilde{R}_3$ is a nonsymmetric relation, that is, there exist $x, y \in X$ with $\mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x)$, which is not antisymmetric and therefore also not perfectly antisymmetric.

One could certainly define other concepts, such as an α-antisymmetry ($|\mu_{\tilde{R}}(x, y) - \mu_{\tilde{R}}(y, x)| \geq \alpha \ \forall x, y \in X$). These concepts would probably be more in line with the basic ideas of fuzzy set theory. Since we will not need this type of definition for our further considerations, we will abstain from any further definition in this direction.

Example 6–9

Let X and Y be defined as in example 6–8. The following relation is then a symmetric relation:

$$\tilde{R}(x, y): \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0 & .1 & 0 & .1 \\ x_2 & .1 & 1 & .2 & .3 \\ x_3 & 0 & .2 & .8 & .8 \\ x_4 & .1 & .3 & .8 & 1 \end{array}$$

Remark 6–1

For max-min compositions, the following properties hold:

1. If $\tilde{R}_1$ is reflexive and $\tilde{R}_2$ is an arbitrary fuzzy relation, then $\tilde{R}_1 \circ \tilde{R}_2 \supseteq \tilde{R}_2$ and $\tilde{R}_2 \circ \tilde{R}_1 \supseteq \tilde{R}_2$.
2. If $\tilde{R}$ is reflexive, then $\tilde{R} \subseteq \tilde{R} \circ \tilde{R}$.
3. If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive relations, so is $\tilde{R}_1 \circ \tilde{R}_2$.
4. If $\tilde{R}_1$ and $\tilde{R}_2$ are symmetric, then $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric if $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$.
5. If $\tilde{R}$ is symmetric, so is each power of $\tilde{R}$.

Transitivity

Definition 6–13

A fuzzy relation $\tilde{R}$ is called (max-min) *transitive* if

$$\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$$

Example 6–10

Let the fuzzy relation $\tilde{R}$ be defined as

$\tilde{R}$:	x_1	x_2	x_3	x_4
x_1	.2	1	.4	.4
x_2	0	.6	.3	0
x_3	0	1	.3	0
x_4	.1	1	1	.1

Then $\tilde{R} \circ \tilde{R}$ is

	x_1	x_2	x_3	x_4
x_1	.2	.6	.4	.2
x_2	0	.6	.3	0
x_3	0	.6	.3	0
x_4	.1	1	.3	.1

Now one can easily see that $\mu_{\tilde{R}\circ\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, y)$ holds for all $x, y \in X$.

Remark 6–2

Combinations of the above properties give some interesting results for max-min compositions:

1. If $\tilde{R}$ is symmetric and transitive, then $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x)$ for all $x, y \in X$.
2. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$.
3. If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive.

The properties mentioned in remarks 6–1 and 6–2 hold for the *max-min composition*. For the *max-prod composition*, property 3 of remark 6–2 is also true but not properties 1 and 3 of remark 6–1 or property 1 of remark 6–2. For the *max-av composition*, properties 1 and 3 of remark 6–1 hold as well as properties 1 and 3 of remark 6–2. Property 5 of remark 6–1 is true for any commutative operator.

6.2 Fuzzy Graphs

It was already mentioned that definitions 6–1 and 6–2 of a fuzzy relation can also be interpreted as defining a fuzzy graph. In order to stay in line with the terminology of traditional graph theory we shall use the following definition of a fuzzy graph.

Definition 6–14

Let E be the (crisp) set of nodes. A *fuzzy graph* is then defined by

$$\tilde{G}(x_i, x_j) = \{((x_i, x_j), \mu_{\tilde{G}}(x_i, x_j)) | (x_i, x_j) \in E \times E\}$$

If $\tilde{E}$ is a fuzzy set, a fuzzy graph would have to be defined in analogy to definition 6–2.

Example 6–11

a. Let $E = \{A, B, C\}$.
   Considering only three possible degrees of membership, graphs could be described as shown in figure 6–1.
b. Let $E = \{x_1, x_2, x_3, x_4\}$; then a fuzzy graph could be described as

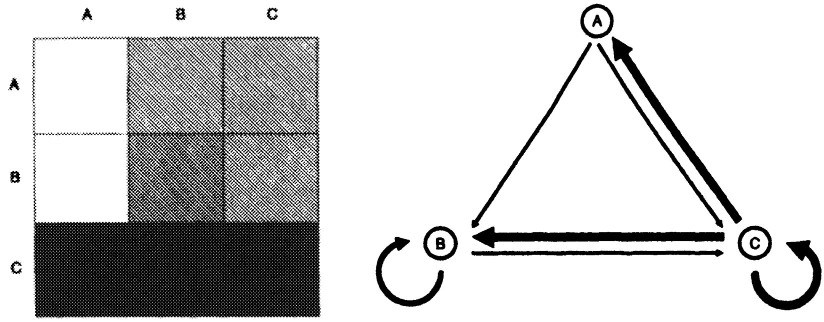


Figure 6–1. Fuzzy graphs.

$$\tilde{G}(x_i, x_j) = \{[(x_1, x_2), .3], [(x_1, x_3), .6], [(x_1, x_1), 1], [(x_2, x_1), .4], [(x_3, x_1), .2], [(x_3, x_2), .5], [(x_4, x_3), .8]\}$$

Example 6–11a shows directed fuzzy binary graphs. Graphs can, of course, also be defined in higher-dimension product spaces. We shall, however, focus our attention on finite undirected binary graphs; that is, we shall assume in the following that the fuzzy relation representing a graph is symmetric. The arcs can then be considered as unordered pairs of nodes. In analogy to traditional graph theory, fuzzy graph theoretic concepts can be defined.

Definition 6–15

$\tilde{H}(x_i, x_j)$ is a *fuzzy subgraph* of $\tilde{G}(x_i, x_j)$ if

$$\mu_{\tilde{H}}(x_i, x_j) \leq \mu_{\tilde{G}}(x_i, x_j) \ \forall (x_i, x_j) \in E \times E$$

$\tilde{H}(x_i, x_j)$ *spans* graph $\tilde{G}(x_i, x_j)$ if the node sets of $\tilde{H}(x_i, x_j)$ and $\tilde{G}(x_i, y_j)$ are equal, that is, if they differ only in their arc weights.

Example 6–12

Let $\tilde{G}(x_i, x_j)$ be defined as in example 6–11b. A spanning subgraph of $\tilde{G}(x_i, x_j)$ is then

$$\tilde{H}(x_i, x_j) = \{[(x_1, x_2), .2], [(x_1, x_3), .4], [(x_3, x_2), .4], [(x_4, x_3), .7]\}$$

Definition 6–16

A *path* in a fuzzy graph $\tilde{G}(x_i, x_j)$ is a sequence of distinct nodes, $x_0, x_1, \ldots, x_n$, such that for all (x_i, x_{i+1}), $\mu_{\tilde{G}}(x_i, x_{i+1}) > 0$. The *strength* of the path is min $\{\mu_{\tilde{G}}(x_i, x_{i+1})\}$ for all nodes contained in the path. The *length* of a path $n > 0$ is the number of nodes contained in the path. Each pair of nodes (x_i, x_{i+1}), $\mu(x_i, x_{i+1}) > 0$ is called an *edge* (arc) of the graph. A path is called a *cycle* if $x_0 = x_n$ and $n \geq 3$.

It would be straightforward to call the length of the shortest path between two nodes of the graph the distance between these nodes. This definition, however, has some disadvantages. It is therefore more reasonable to define the distance between two nodes as follows [Rosenfeld 1975, p. 58]:

Definition 6–17

The μ-*length* of a path $p = x_0, \ldots, x_n$ is equal to

$$L(p) = \sum_{i=1}^{n} \frac{1}{\mu(x_i, x_{i+1})}$$

The μ-*distance* $d(x_i, x_j)$ between two nodes x_i, x_j is the smallest μ-length of any path from x_i to x_j, $x_i, x_j \in \tilde{G}$.

It can then be shown [see Rosenfeld 1975, p. 88] that $d(x_i, x_j)$ is a metric (in undirected graphs!).

Definition 6–18

Two nodes that are joined by a path are called *connected nodes.*

Connectedness is a relation that is also transitive.

Definition 6–19

A fuzzy graph is a *forest* if it has no cycles; that is, it is an acyclic fuzzy graph. If the fuzzy forest is connected, it is called a *tree*. (A fuzzy graph that is a forest has to be distinguished from a fuzzy graph that is a fuzzy forest.) The latter shall not be discussed here [see Rosenfeld 1975, p. 92].

Example 6–13

The fuzzy graphs shown in figure 6–2 are forests. The graphs shown in figure 6–3 are not.

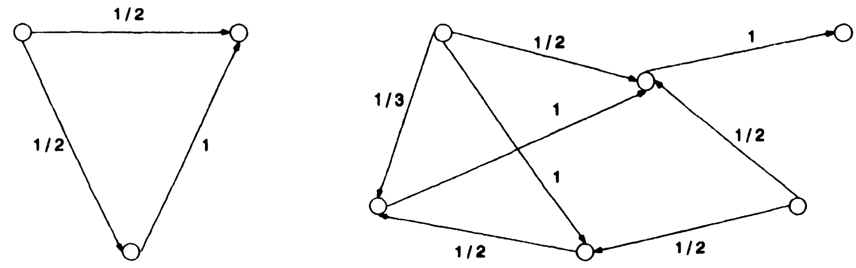


Figure 6–2. Fuzzy forests.

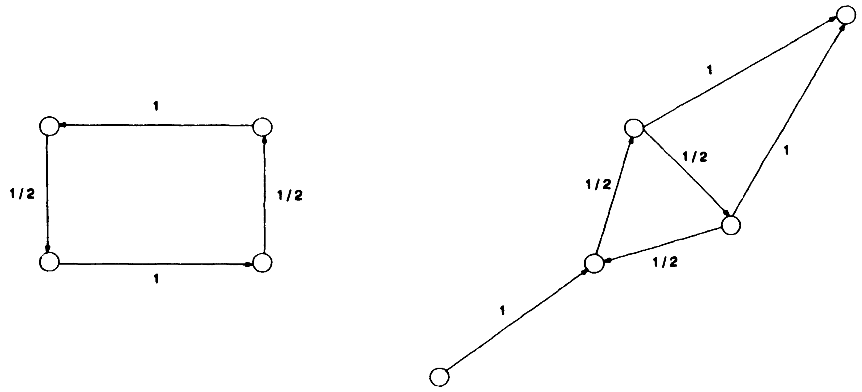


Figure 6–3. Graphs that are not forests.

6.3 Special Fuzzy Relations

Relations that are of particular interest to us are fuzzy relations that pertain to the similarity of fuzzy sets and those that order fuzzy sets. All of the relations discussed below are reflexive, that is, $\mu_{\tilde{R}}(x, x) = 1\ \forall x \in X$ [Zadeh 1971], and they are max-min transitive, that is, $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$ or $\mu_{\tilde{R}}(x, z) \geq \min\ \{\mu_{\tilde{R}}(x, y), \mu_{\tilde{R}}(y, z)\}$ $\forall x, y, z \in X$. It should be noted that other kinds of transitivities have been defined [see Bezdek and Harris 1978]. These, however, will not be discussed here. The main difference between similarity relations and order relations is the property of symmetry or antisymmetry, respectively.

Definition 6–20

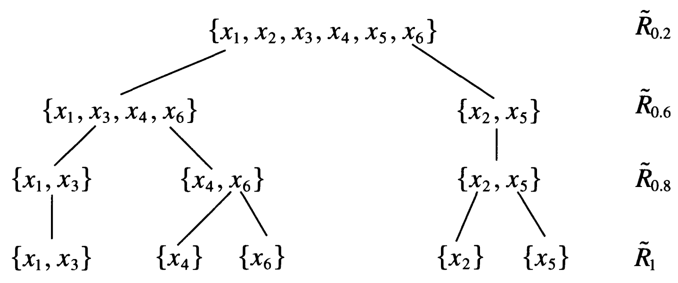
A *similarity relation* is a fuzzy relation $\mu_s(\cdot)$ that is reflexive, symmetrical, and max-min transitive.

Example 6–14

The following relation is a similarity relation [Zadeh 1971]:

		x_1	x_2	x_3	x_4	x_5	x_6
	x_1	1	.2	1	.6	.2	.6
	x_2	.2	1	.2	.2	.8	.2
$\tilde{R}_s$:	x_3	1	.2	1	.6	.2	.6
	x_4	.6	.2	.6	1	.2	.8
	x_5	.2	.8	.2	.2	1	.2
	x_6	.6	.2	.6	.8	.2	1

A similarity relation of a finite number of elements can also be represented by a *similarity tree*, similar to a dendogram. In this tree, each level represents an α-cut (α-level set) of the similarity relation. For the above similarity relation, the similarity tree is shown below. The sets of elements on specific α-levels can be considered as similarity classes of α-level.



The properties of a similarity relation as defined in definition 6–20 are rather restrictive and not quite in accordance with fuzzy set thinking: Reflexitivity could be considered as being too restrictive and hence weakened by substituting these requirements by ε-reflexitivity or weak reflexitivity (cf. definition 6–10). The max-min transitivity can be replaced by any max-* transitivity listed in definition 6–10 or in remark 6–1.

We shall now turn to fuzzy order relations: As already mentioned, similarity relations and order relations are primarily distinguished by their degree of symmetry. Roughly speaking, similarity relations are fuzzy relations that are reflexive, (max-min) transitive, and symmetrical; order relations, however, are not symmetrical. To be more precise, even different kinds of fuzzy order relations differ by their degree of symmetry.

Definition 6–21

A fuzzy relation that is (max-min) transitive and reflexive is called a *fuzzy preorder relation.*

Definition 6–22

A fuzzy relation that is (min-max) transitive, reflexive, and antisymmetric is called a *fuzzy order relation.* If the relation is perfectly antisymmetrical, it is called a *perfect fuzzy order relation* [Kaufmann 1975, p. 113]. It is also called a *fuzzy partial order relation* [Zadeh 1971].

Definition 6–23

A *total fuzzy order relation* [Kaufmann 1975, p. 112] or a *fuzzy linear ordering* [Dubois and Prade 1980a, p. 82; Zadeh 1971] is a fuzzy order relation such that $\forall x, y \in X; x \neq y$ either $\mu_{\tilde{R}}(x, y) > 0$ or $\mu_{\tilde{R}}(y, x) > 0$.

Any α-cut of a fuzzy linear order is a *crisp linear order.*

Example 6–15

$\tilde{R}$:	y_1	y_2	y_3	y_4
x_1	.7	.4	.8	.8
x_2	0	1	0	.2
x_3	0	.6	0	.4
x_4	0	0	0	.7

$\tilde{R}$ is a total fuzzy order relation.

Fuzzy order relations play a very important role in models for decision making in fuzzy environments. We will therefore elaborate on some particularly interesting properties in the second volume, and we shall also discuss some additional concepts in this context. Some of the properties of the special fuzzy relations defined in this chapter are summarized in table 6–1.

Table 6–1. Properties of fuzzy relations.

	Reflexivity	*Transitivity*	*Anti-symmetry*	*Perfect anti-symmetry*	*Linearity*	*Symmetry*
Fuzzy preorder	×	×				
Similarity relation	×	×				×
Fuzzy order relation	×	×	×			
Perfect fuzzy order relation	×	×		×		
Total (linear) fuzzy order relation	×	×		×	×	

Exercises

1. Given an example for the membership function of the fuzzy relation $\tilde{R}$: = "considerable smaller than" in $R \times R$. Restrict $\tilde{R}$ to the first ten natural numbers and define the resulting matrix.
2. Let the two fuzzy sets $\tilde{A}$ and $\tilde{B}$ be defined as

$$\tilde{A} = \{(0, .2), (1, .3), (2, .4), (3, .5)\}$$

$$\tilde{B} = \{(0, .5), (1, .4), (2, .3), (3, .0)\}.$$

Is the following set a fuzzy relation on $\tilde{A}$ and $\tilde{B}$?

$$\{((0, 0), .2), ((0, 2), .2), ((2, 0), .2)\}$$

Give an example of a fuzzy relation on $\tilde{A}$ and $\tilde{B}$.

3. Consider the following matrix defining a fuzzy relation $\tilde{R}$ on $\tilde{A} \times \tilde{B}$.

$\tilde{R}$:	y_1	y_2	y_3	y_4	y_5
x_1	.5	0	1	.9	.9
x_2	1	.4	.5	.3	.1
x_3	.7	.8	0	.2	.6
x_4	.1	.3	.7	1	0

Given the first and the second projection with $\mu_{\tilde{R}^{(1)}}(x)$ and $\mu_{\tilde{R}^{(2)}}(y)$ and the cylindrical extensions of the projection relations with $\mu_{\tilde{R}^{(1)}L}$ and $\mu_{\tilde{R}^{(2)}L}$.

4. Compose the following two fuzzy relations $\tilde{R}_1$ and $\tilde{R}_2$ by using the
= max-min composition,
= max-prod. composition, and
= max-av. composition.

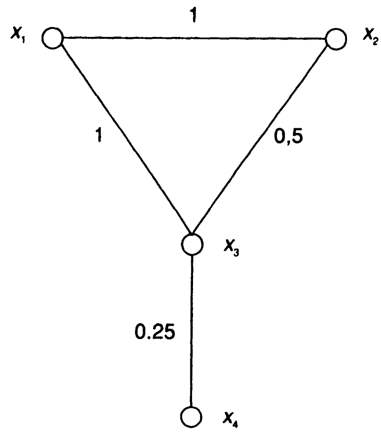
R_1	y_1	y_2	y_3	y_4
x_1	.3	0	.7	.3
x_2	0	1	.2	0

$\tilde{R}_2$	z_1	z_2	z_3
y_1	1	0	1
y_2	0	.5	.4
y_3	.7	.9	.6
y_4	0	0	0

5. Discuss the reflexivity properties of the following fuzzy relation:

$\tilde{R}$	x_1	x_2	x_3
x_1	1	.7	.3
x_2	.4	.5	.8
x_3	.7	.5	1

6. Give an example for a reflexive transitive relation and verify remark 6–2.2.
7. Consider the following fuzzy graph $\tilde{G}$:



Give an example for a spanning subgraph of $\tilde{G}$!
Give all paths from x_1 to x_4 and determine their strengths and their μ lengths.
Is the above graph a forest or a tree?

8. In example 6–2, two relations are defined without specifying for which numerical values of $\{x_i\}$, $\{y_i\}$ the relations are good interpretations of the verbal relations. Give examples of numerical vectors for $\{x_i\}$ and $\{y_i\}$ such that the relations $\tilde{R}$ and $\tilde{Z}$, respectively (in the matrixes), would express the verbal description.

7 FUZZY ANALYSIS

7.1 Fuzzy Functions on Fuzzy Sets

A fuzzy function is a generalization of the concept of a classical function. A classical function f is a mapping (correspondence) from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be considered to be fuzzy rather than crisp. Therefore different "degrees" of fuzzification of the classical notion of a function are conceivable.

1. There can be a crisp mapping from a fuzzy set that carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.
2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. This we shall call a *fuzzy function*. These are called "fuzzifying functions" by Dubois and Prade [1980a, p. 106].
3. Ordinary functions can have fuzzy properties or be constrained by fuzzy constraints.

Naturally, hybrid types can be considered. We shall focus our considerations, however, only on frequently used pure cases.

H.-J. Zimmermann, *Fuzzy Set Theory - and Its Applications*

Definition 7–1 [Dubois and Prade 1980a; Negoita and Ralescu 1975]

A *classical function* $f: X \to Y$ maps from a *fuzzy domain* $\tilde{A}$ in X into a fuzzy range $\tilde{B}$ in Y iff

$$\forall x \in X, \mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x)$$

Given a classical function $f: X \to Y$ and a fuzzy domain $\tilde{A}$ in X, the extension principle (chapter 5.1) yields the fuzzy range $\tilde{B}$ with the membership function

$$\mu_{\tilde{B}}(y) = \sup_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x)$$

Hence f is a function according to definition 7–1.

Example 7–1

Let X be the set of temperatures, Y the possible demands for energy of households, $\tilde{A}$ the fuzzy set "low temperatures," and $\tilde{B}$ the fuzzy set "high energy demands." The assignment "low temperatures" $\to$ "high energy demands" is then a fuzzy function, and the additional constraint in definition 7–1 means "the lower the temperatures, the higher the energy demands."

The correspondence between a fuzzy function and a fuzzy relation becomes even more obvious when looking at the following definition.

Definition 7–2

Let X and Y be universes and $\tilde{P}(Y)$ the set of all fuzzy sets in Y (power set).

$$\tilde{f}: X \to \tilde{P}(Y) \text{ is a mapping}$$
$$\tilde{f} \text{ is a } \textit{fuzzy function} \text{ iff}$$
$$\mu_{\tilde{f}(x)}(y) = \mu_{\tilde{R}}(x, y), \forall (x, y) \in X \times Y$$

where $\mu_{\tilde{P}}(x, y)$ is the membership function of a fuzzy relation.

Example 7–2

a. Let X be the set of all workers of a plant, $\tilde{f}$ the daily output, and y the number of processed work pieces. A fuzzy function could then be

$$\tilde{f}(x) = y$$

b. $\tilde{a}, \tilde{b} \in \mathscr{L}(\mathbb{R})$
$X = \mathbb{R}$
$\tilde{f}: x \to \tilde{a}x \oplus \tilde{b}$ is a fuzzy function.

c. X = set of all one-mile runners.
$\tilde{f}$ = possible record times.
$\tilde{f}(x) = \{y | y\text{: achieved record times}\}$.

7.2 Extrema of Fuzzy Functions

Traditionally, an extremum (maximum or minimum) of a crisp function f over a given domain D is attained at a precise point x_0. If the function f happens to be the objective function of a decision model, possibly constrained by a set of other functions, then the point x_0 at which the function attains the optimum is generally called the optimal decision; that is, in classical theory there is an almost unique relationship between the extremum of the objective function and the notion of the optimal decision of a decision model.

In models in which fuzziness is involved, this unique relationship no longer exists. The extremum of a function or the optimum of a decision model can be interpreted in a number of ways: In decision models the "optimal decision" is often considered to be the crisp set, D_m, that contains those elements of the fuzzy set "decision" attaining the maximum degree of membership [Bellman and Zadeh 1970, p. 150]. We shall discuss this concept in more detail in chapter 13.

The notion of an "optimal decision" as mentioned above corresponds to the concept of a "maximizing set" when considering functions in general.

Definition 7–3 [Zadeh 1972]

Let f be a real-valued function in X. Let f be bounded from below by $\inf(f)$ and from above by $\sup(f)$. The fuzzy set $\tilde{M} = \{(x, \mu_{\tilde{M}}(x)\}, x \in X$ with

$$\mu_{\tilde{M}}(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)}$$

is then called the *maximizing set* (see figure 7–1).

Example 7–3

$$f(x) = \sin x$$

$$\mu_{\tilde{M}}(x) = \frac{\sin x - \inf(\sin)}{\sup(\sin) - \inf(\sin)} = \frac{\sin x - (-1)}{1 - (-1)}$$

$$= \frac{\sin x + 1}{2} = \frac{1}{2}\sin x + \frac{1}{2}$$

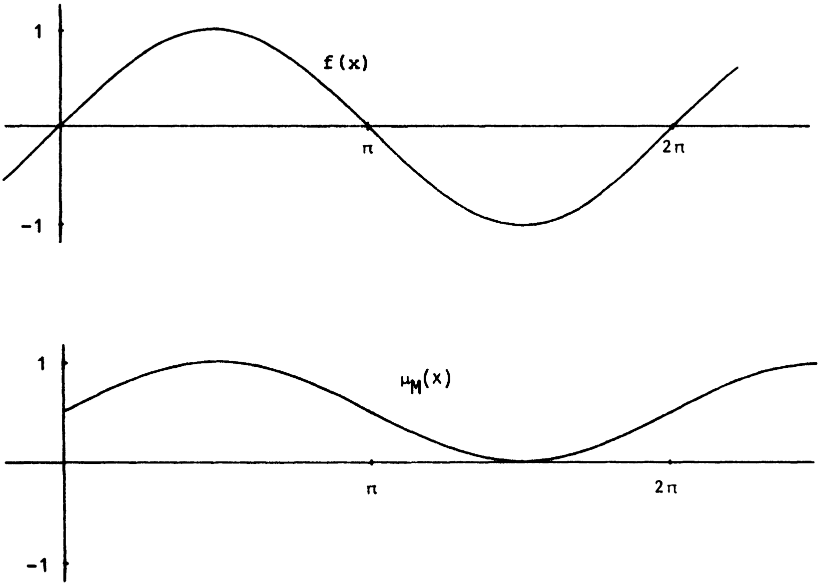


Figure 7–1. Maximizing set.

In definition 7–3, f is a crisp real-valued function, similar to the membership function of the fuzzy set "decision," and the maximizing set provides information about the neighborhood of the extremum of the function f, the domain of which is also crisp. The case in which the domain of f is also fuzzy will be considered in chapter 13.

Let us now consider the extrema of fuzzy functions according to definition 7–2, in which they are defined over a crisp domain: Since a fuzzy function $f(x)$ is a fuzzy set, say in $\mathbb{R}$, the maximum will generally not be a point in $\mathbb{R}$ but also a fuzzy set, which we shall call the "fuzzy maximum of $f(x)$." A straightforward approach is to define an extended max operation in analogy to the other extended operations defined in chapter 5. Max and min are increasing operations in $\mathbb{R}$. The maximum or minimum, respectively, of n fuzzy numbers, denoted by mãx $(\tilde{M}_1, \ldots, \tilde{M}_n)$ and mĩn $(\tilde{M}_1, \ldots, \tilde{M}_n)$, is again a fuzzy number. Dubois and Prade [1980a, p. 58] present rules for computing mãx and mĩn and also comment on the properties of mãx and mĩn. The reader is referred to the above reference for further details.

Definition 7–4

Let $\tilde{f}(x)$ be a fuzzy function from X to $\mathbb{R}$, defined over a crisp and finite domain D. The *fuzzy maximum* of $\tilde{f}(x)$ is then defined as

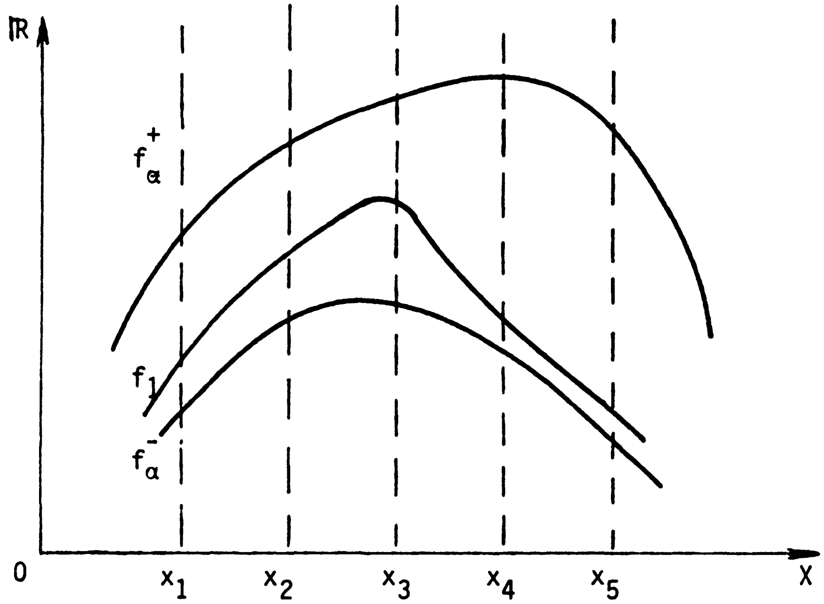


Figure 7–2. A fuzzy function.

$$\tilde{M} = \max_{x \in D} \tilde{f}(x) = \{(\sup \tilde{f}(x), \mu_{\tilde{M}}(x)) \mid x \in D\}$$

For $|D| = n$, the membership function of $\tilde{\max} \tilde{f}(x)$ is given by

$$\mu_{\tilde{M}}(x) = \min_{j=1,\ldots,n} \mu_{\tilde{f}(x_j)}(\tilde{f}(x_j)), \quad f(x) \in D$$

Example 7–4 [Dubois and Prade 1980a, p. 105]

Let $\tilde{f}(x)$ be a fuzzy function from $\mathbb{R}$ to $\mathbb{R}$ such that, for any x, $\tilde{f}(x)$ is a triangular fuzzy number. The domain $D = \{x_1, x_2, x_3, x_4, x_5\}$. Figure 7–2 sketches such a function by showing for the domain D "level curves" of $\tilde{f}(x)$: f_1 is the curve for which $\mu_{\tilde{f}(x)}(f_1(x)) = 1$, and for f_α^+ and f_α^-, respectively,

$$\mu_{\tilde{f}(x)}(f_\alpha^-(x)) = \mu_{\tilde{f}(x)}(f_\alpha^+(x)) = \alpha$$

The triangular fuzzy numbers representing the function $\tilde{f}(x)$ at $x = x_1, x_2, x_3, x_4$, and x_5 are shown in figure 7–3.

We can make the following observation: Since the level curves in figure 7–2 are not parallel to each other, their maxima are attained at different x_i: max

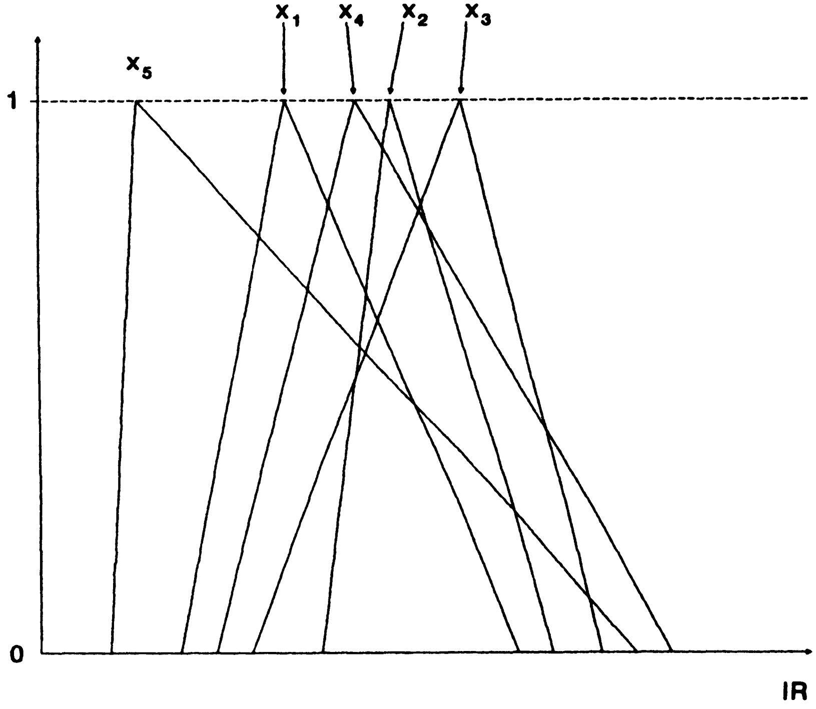


Figure 7–3. Triangular fuzzy numbers representing a fuzzy function.

$f_\alpha^+ = f_\alpha^+(x_4)$, $\max f_1(x) = f_1(x_3)$, and $\max f_\alpha^-(x) = f_\alpha^-(x_2)$. Thus x_1 and x_5 do certainly not "belong" to the maximum of $f(x)$. We can easily determine the fuzzy set "maximum of $\tilde{f}(x)$" as defined in definition 7–4 by looking at figure 7–4 and observing that, for

$$\alpha \in [0, \alpha^-]: f^-(x_2) \geq f_\alpha^-(x_i)\ \forall i$$
$$\alpha \in [\alpha^-, 1]: f^-(x_3) \geq f_\alpha^-(x_i)\ \forall i$$
$$\alpha \in [\alpha^+, 1]: f^+(x_3) \geq f_\alpha^+(x_i)\ \forall i$$
$$\alpha \in [0, \alpha^+]: f^+(x_4) \geq f_\alpha^+(x_i)\ \forall i$$

with α^- and α^+ such that $f_\alpha^-(x_2) = f_\alpha^-(x_3)$ and $f_\alpha^+(x_4) = f_\alpha^+(x_3)$, respectively.

The maximum of $\tilde{f}(x)$ is therefore

$$\tilde{M} = \{(x_2, \alpha^-), (x_3, 1), (x_4, \alpha^+)\}$$

This set is indicated in figure 7–4 by the dashed line.

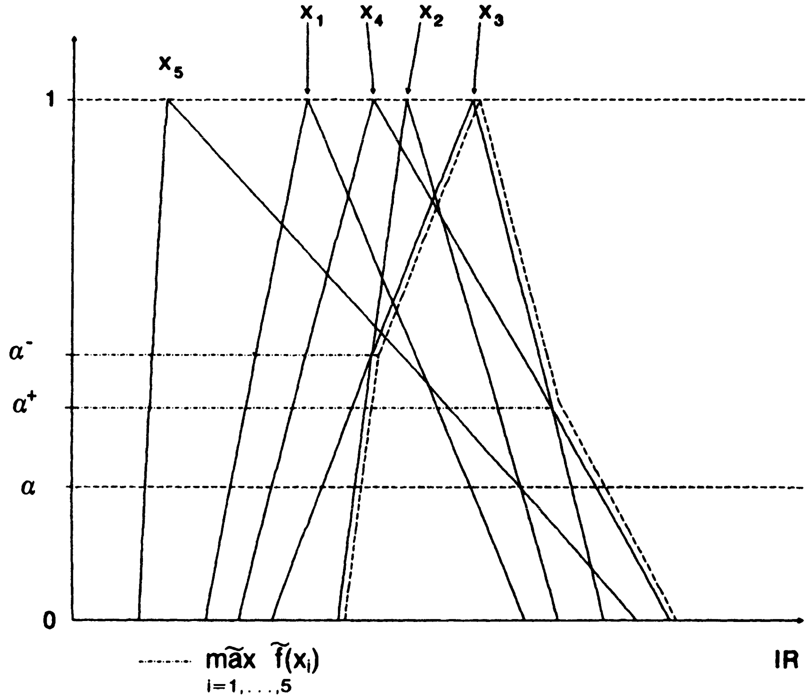


Figure 7–4. The maximum of a fuzzy function.

Dubois and Prade [1980a, p. 101] suggest additional possible interpretations of fuzzy extrema, which might be very appropriate in certain situations. However, we shall not discuss them here and rather shall proceed to consider possible notions of the integral of a fuzzy set or a fuzzy function.

7.3 Integration of Fuzzy Functions

Quite different suggestions have been made to define fuzzy integrals, integrals of fuzzy functions, and integrals of crisp functions over fuzzy domains or with fuzzy ranges.

One of the first concepts of a fuzzy integral was put forward by Sugeno [1972, 1977], who considered fuzzy measures and suggested a definition of a fuzzy integral that is a generalization of Lebesque integrals: "From the viewpoint of functionals, fuzzy integrals are merely a kind of nonlinear functionals (precisely speaking, monotonous functionals), while Lebesque integrals are linear ones" [Sugeno 1977, p. 92].

We shall focus our attention on approaches along the line of Riemann integrals. The main references for the following are Dubois and Prade [1980a, 1982b], Aumann [1965], and Nguyen [1978].

The classical concept of integration of a real-valued function over a closed interval can be generalized in four ways: The function can be a fuzzy function that is to be integrated over a crisp interval, or it can be integrated over a fuzzy interval (that is, an interval with fuzzy foundations). Alternatively, we may consider integrating a fuzzy function as defined in definitions 7–1 or 7–2 over a crisp or a fuzzy interval.

7.3.1 Integration of a Fuzzy Function over a Crisp Interval

We shall now consider a fuzzy function $\tilde{f}$, according to definition 7–2, which shall be integrated over the crisp interval $[a, b]$. The fuzzy function $\tilde{f}(x)$ is supposed to be a fuzzy number, that is, a piecewise continuous convex normalized fuzzy set on $\mathbb{R}$.

We shall further assume that the α-level curves (see definition 2.3) $\mu_{\tilde{f}(x)}(y) = \alpha$ for all $\alpha \in [0, 1]$ and α and x as parameters have exactly two continuous solutions, $y = f_\alpha^+(x)$ and $y = f_\alpha^-(x)$, for $\alpha \neq 1$ and only one for $\alpha = 1$. f_α^+ and f_α^- are defined such that

$$f_{\alpha'}^+(x) \geq f_\alpha^+(x) \geq f(x) \geq f_\alpha^- \geq f_{\alpha'}^-$$

for all $\alpha' \geq \alpha$.

The integral of any continuous α-level curve of $\tilde{f}$ over $[a, b]$ always exists.

One may now define the integral $\tilde{I}(a, b)$ of $\tilde{f}(x)$ over $[a, b]$ as a fuzzy set in which the degree of membership α is assigned to the integral of any α-level curve of $\tilde{f}(x)$ over $[a, b]$.

Definition 7–5

Let $f(x)$ be a fuzzy function from $[a, b] \subseteq \mathbb{R}$ to $\mathbb{R}$ such that $\forall x \in = [a, b]\ \tilde{f}(x)$ is a fuzzy number and $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are α-level curves as defined above. The *integral* of $\tilde{f}(x)$ over $[a, b]$ is then defined to be the fuzzy set

$$\tilde{I}(a, b) = \left\{\left(\int_a^b f_\alpha^-(x)\,dx + \int_a^b f_\alpha^+(x)\,dx, \alpha\right)\right\}$$

This definition is consistent with the extension principle according to which

$$\mu_{\int_a^b f}(y) = \sup_{\substack{g \in y \\ y = \int_a^b g}} \inf_{x \in [a,b]} \mu_{f(x)}(g(x)), \quad y \in \mathbb{R}$$

where $y = \{g\colon [a, b] \to \mathbb{R} \mid g \text{ integrable}\}$ see Dubois and Prade [1980a, p. 107; 1982, p. 5]).

The determination of the integral $\tilde{I}(a, b)$ becomes somewhat easier if the fuzzy function is assumed to be of the *LR* type (see definition 5–6). We shall therefore assume that $\tilde{f}(x) = (f(x), s(x), t(x))_{LR}$ is a fuzzy number in *LR* representation for all $x \in [a, b]$. f, s, and t are assumed to be positive integrable functions on $[a, b]$. Dubois and Prade [1980a, p. 109] have shown that under these conditions

$$\tilde{I}(a,b) = \left(\int_a^b f(x)dx, \int_a^b s(x)dx, \int_a^b t(x)dx\right)_{LR}$$

It is then sufficient to integrate the mean value and the spread functions of $\tilde{f}(x)$ over $[a, b]$, and the result will again be an *LR* fuzzy number.

Example 7–5

Consider the fuzzy function $\tilde{f}(x) = (f(x), s(x), t(x))_{LR}$ with the mean function $f(x) = x^2$, the spread functions $s(x) = x/4$, and

$$t(x) = \frac{x}{2}$$

$$L(x) = \frac{1}{1+x^2}$$

$$R(x) = \frac{1}{1+2|x|}$$

Determine the integral from $a = 1$ to $b = 4$, that is, compute $\int_1^4 f$.

According to the above formula, we compute

$$\int_a^b f(x)dx = \int_1^4 x^2\,dx = 21$$

$$\int_a^b s(x)dx = \int_1^4 \frac{x}{4}dx = 1.875$$

$$\int_a^b t(x)dx = \int_1^4 \frac{x}{2}dx = 3.75$$

This yields the fuzzy number $\tilde{I}(a, b) = (21, 1.875, 3.75)_{LR}$ as the value of the fuzzy integral.

Some Properties of Integrals of Fuzzy Functions. Let A_α be the α-level set of the fuzzy set $\tilde{A}$. The support $S(\tilde{A})$ of $\tilde{A}$ is then $S(\tilde{A}) = \bigcup_{\alpha\in[0,1]} A_\alpha$. The fuzzy set $\tilde{A}$ can now be written as

$$\tilde{A} = \bigcup_{\alpha\in[0,1]} \alpha A_\alpha = \bigcup_{\alpha\in[0,1]} \{(x, \mu_{\alpha A_\alpha}(x) | x \in A_\alpha)\}$$

where

$$\mu_{\alpha A_\alpha}(x) = \begin{cases} \alpha & \text{for } x \in A_\alpha \\ 0 & \text{for } x \notin A_\alpha \end{cases}$$

(see Nguyen [1978, p. 369]).

Let $\tilde{A}$ represent a fuzzy integral, that is,

$$\tilde{A} = \int_I \tilde{f}$$

then

$$\int_I \tilde{f} = \bigcup_{\alpha\in[0,1]} \alpha\left(\int_I \tilde{f}\right)_\alpha$$
$$= \bigcup_{\alpha\in[0,1]} \alpha\left(\int_I \tilde{f}_\alpha\right)$$

Definition 7–6 [Dubois and Prade 1982a, p. 6]

$\int_I \tilde{f}$ satisfies the *commutativity condition*

$$\text{iff } \forall\alpha \in [0,1] \left(\int_I \tilde{f}\right)_\alpha = \int_I \tilde{f}_\alpha$$

Dubois and Prade [1982a, p. 6] have proved the following properties of fuzzy integrals, which are partly a straightforward analogy of crisp analysis.

Theorem 7–1

Let $\tilde{f}$ be a fuzzy function; then

$$\int_I \tilde{f} = \int_a^b \tilde{f} = -\int_b^a \tilde{f}$$

where the fuzzy integrals are fuzzy sets with the membership functions

$$\mu_{-\int_a^b f}(u) = \mu_{\int_a^b f}(-u)\ \forall u$$

Theorem 7–2

Let I and I' be two adjacent intervals $I = [a, b]$, $I' = [b, c]$ and a fuzzy function $\tilde{f}$: $[a, c] \rightarrow \tilde{P}(\mathbb{R})$. Then

$$\int_a^c \tilde{f} = \int_a^b \tilde{f} \oplus \int_b^c \tilde{f}$$

where $\oplus$ denotes the extended addition of fuzzy sets, which is defined in analogy to the subtraction of fuzzy numbers (see chapter 5).

Let $\tilde{f}$ and $\tilde{g}$ be fuzzy functions. Then $\tilde{f} \oplus \tilde{g}$ is pointwise defined by

$$(\tilde{f} \oplus \tilde{g})(u) = \tilde{f}(u) \oplus \tilde{g}(u), \quad u \in X$$

(This is a straightforward application of the extension principle from chapter 5.1.)

Theorem 7–3

Let $\tilde{f}$ and $\tilde{g}$ be fuzzy functions whose supports are bounded. Then

$$\int_I (\tilde{f} \oplus \tilde{g}) \supseteq \int_I \tilde{f} \oplus \int_I \tilde{g} \tag{7.1}$$

$$\int_I (\tilde{f} \oplus \tilde{g}) = \int_I \tilde{f} \oplus \int_I \tilde{g} \tag{7.2}$$

iff the *commutativity condition* is satisfied for $\int \tilde{f}$ and $\int_I \tilde{g}$.

7.3.2 Integration of a (Crisp) Real-Valued Function over a Fuzzy Interval

We now consider a case for which Dubois and Prade [1982a, p. 106] proposed a quite interesting solution: A fuzzy domain $\mathcal{F}$ of the real line $\mathbb{R}$ is assumed to be bounded by two normalized convex fuzzy sets, the membership functions of which are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, respectively. (See figure 7–5.) $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ can be interpreted as the degrees (of confidence) to which x can be considered a lower or upper bound of $\mathcal{F}$. If $\underline{a}_0$ and $\overline{b}_0$ are the lower/upper limits of the supports of $\tilde{a}$ or $\tilde{b}$, then a_0 or b_0 are related to each other by $\underline{a}_0 = \inf S(\tilde{a}) \leq \sup S(\tilde{b}) = \overline{b}_0$.

Definition 7–7

Let f be a real-valued function that is integrable in the interval $J = [a_0, b_0]$; then according to the extension principle, the membership function of the integral $\int_{\mathcal{F}} f$ is given by

$$\mu_{\int_{\mathcal{F}} f}(z) = \sup_{x,y \in J} \min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y))$$

$$z = \int_x^y f$$

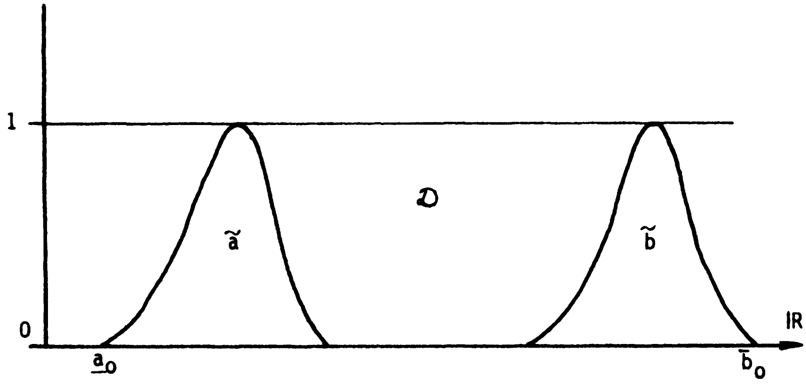


Figure 7–5. Fuzzily bounded interval.

Let $F(x) = \int_c^x f(y)\,dy$, $c \in J$ (F is the antiderivative of f). Then, using the extension principle again, the membership function of $F(\tilde{a})$, $\tilde{a} \in \tilde{P}(\mathbb{R})$, is given by

$$\mu_{f(\tilde{a})}(z) = \sup_{x:z=F(x)} \mu_{\tilde{a}}(x)$$

Proposition 7–1 [Dubois and Prade 1982b, p. 106]

$$\int_{\tilde{\mathfrak{D}}} f = F(\tilde{b}) \ominus F(\tilde{a})$$

where $\ominus$ denotes the extended subtraction of fuzzy sets.

Proofs of proposition 7–1 and of the following propositions can be found in Dubois and Prade [1982b, pp. 107–109].

A possible interpretation of proposition 7–1 is as follows: If $\tilde{a}$ and $\tilde{b}$ are normalized convex fuzzy sets, then $\int_{\tilde{\mathfrak{D}}} f$ is the interval between "worst" and "best" values for different levels of confidence indicated by the respective degrees of membership (see also Dubois and Prade [1988a, pp. 34–36]).

Example 7–6

Let

$$\tilde{a} = \{(4, .8), (5, 1), (6, .4)\}$$
$$\tilde{b} = \{(6, .7), (7, 1), (8, .2)\}$$
$$f(x) = 2, \quad x \in [a_0, b_0] = [4, 8]$$

Then

$$\int_{\tilde{\mathfrak{D}}} f(x)\,dx = \int_{\tilde{a}}^{\tilde{b}} 2dx = 2x\,|_{\tilde{a}}^{\tilde{b}}$$

The detailed computational results are:

(a, b)	$\int_a^b 2dx$	$\min(\mu_x(a), \mu_x(b))$
(4, 6)	4	.7
(4, 7)	6	.8
(4, 8)	8	.2
(5, 6)	2	.7
(5, 7)	4	1.0
(5, 8)	6	.2
(6, 6)	0	.4
(6, 7)	2	.4
(6, 8)	4	.2

Hence choosing the maximum of the membership values for each value of the integral yields $\int_{\tilde{\mathfrak{D}}} f = \{(0, .4), (2, .7), (4, 1), (6, .8), (8, .2)\}$.

Some properties of the integral discussed above are listed in propositions 7–2 to 7–4 below. Their proofs, as well as descriptions of other approaches to "fuzzy integration," can again be found in Dubois and Prade [1982a, pp. 107–108].

Proposition 7–2

Let f and g be two functions $f, g\colon I \to \mathbb{R}$, integrable on I. Then

$$\int_{\tilde{a}}^{\tilde{b}} (f+g) \subseteq \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$$

where $\oplus$ denotes the extended addition (see chapter 5).

Example 7–7

Let

$$f(x) = 2x - 3$$

$$g(x) = -2x + 3$$

$$\tilde{a} = \{(1, .8), (2, 1), (3, .4)\}$$

$$\tilde{b} = \{(3, .7), (4, 1), (5, .3)\}$$

So

$$\int_a^b f(x)dx = [x^2 - 3x]_a^b$$

$$\int_a^b g(x)dx = [-x^2 + 5x]_a^b$$

$$\int_a^b f(x) + g(x)dx = [2x]_a^b$$

In analogy to example 7–6, we obtain

$$\int_{\tilde{a}}^{\tilde{b}} f = \{(0,.4),(2,.7),(4,.4),(6,1),(10,.3),(12,.3)\}$$

$$\int_{\tilde{a}}^{\tilde{b}} g = \{(-6,.3),(-4,.3),(-2,.1),(0,.8),(2,.7)\}$$

Applying the formula for the extended addition according to the extension principle (see section 5.3) yields

$$\int_{\tilde{a}}^{\tilde{b}} f + \int_{\tilde{a}}^{\tilde{b}} g = \{(-6,.3),(-4,.3),(-2,.4),(0,.7),(2,.7),(4,.1),(6,.8),$$
$$(8,.7),(10,.3),(12,.3),(14,.3)\}$$

Similarly to example 7–6, we compute

$$\int_{\tilde{a}}^{\tilde{b}} (f + g) = \{(0,.4),(2,.7),(4,1),(6,.8),(8,.3)$$

Now we can easily verify that

$$\int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g \supseteq \int_{\tilde{a}}^{\tilde{b}} (f + g)$$

Proposition 7–3

If f, g: $I \to R^+$ or f, g: $I \to R^-$,
then equality holds:

$$\int_{\tilde{a}}^{\tilde{b}} (f + g) = \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$$

Proposition 7–4

Let $\tilde{\mathfrak{D}} = (\tilde{a}, \tilde{b})$, $\tilde{\mathfrak{D}}' = (\tilde{a}, \tilde{c})$, and $\tilde{\mathfrak{D}}'' = (\tilde{c}, \tilde{b})$. Then the following relationships hold:

$$\int_{\tilde{\mathfrak{D}}} f \subseteq \int_{\tilde{\mathfrak{D}}} f_1 \oplus \int_{\tilde{\mathfrak{D}}} f_2 \tag{7.3}$$

$$\int_{\tilde{\mathfrak{D}}} f = \int_{\tilde{\mathfrak{D}}} f_1 \oplus \int_{\tilde{\mathfrak{D}}^1} f_2 \quad \text{iff } \tilde{c} \in \mathbb{R} \tag{7.4}$$

7.4 Fuzzy Differentiation

In analogy to integration, differentiation can be extended to fuzzy mathematical structures.

The results will, of course, depend on the type of function considered. In terms of section 7.1, we will focus our attention on functions that are not fuzzy themselves but that only "carry" the possible fuzziness of their arguments. Differentiation of fuzzy functions is considered by Dubois and Prade [1980a, p. 116; 1982b, p. 227].

Here we shall consider only differentiation of a differentiable function f: $\mathbb{R} \subseteq [a, b] \to \mathbb{R}$ at a "fuzzy point." A "fuzzy point" $\tilde{X}_0$ [Dubois and Prade 1982b, p. 225] is a convex fuzzy subset of the real line $\mathbb{R}$ (see definition 2–4).

In the following, fuzzy points will be considered for which the support is contained in the interval $[a, b]$, that is, $S(\tilde{x}) \subseteq [a, b]$.

Such a fuzzy point can be interpreted as the possibility distribution of a point x whose precise location is only approximately known.

The uncertainty of the knowledge about the precise location of the point induces an uncertainty about the derivative $f'(x)$ of a function $f(x)$ at this point. The derivative might be the same for several x belonging to $[a, b]$. The possibility of $f'(\tilde{X}_0)$ is therefore defined [Zadeh 1078] to be the supremum of the values of the possibilities of $f'(x) = t$, $x \in [a, b]$.

The "derivative" of a real-valued function at a fuzzy point can be interpreted as the fuzzy set $f'(\tilde{X}_0)$, the membership function of which expresses the degree to which a specific $f'(x)$ is the first derivative of a function f at point $\tilde{X}_0$.

Definition 7–8

The membership function of the fuzzy set "*derivative* of a real-valued function at a fuzzy point $\tilde{X}_0$" is defined by the extension principle as

$$\mu_{f'(\tilde{X}_0)}(y) = \sup_{x \in f'^{-1}(y)} \mu_{\tilde{X}_0}(x)$$

where $\tilde{X}_0$ is the fuzzy number that characterizes the fuzzy location.

Example 7–8

Let

$$f(x) = x^3$$
$$\tilde{X}_0 = \{(-1, .4), (0, 1), (1, .6)\}$$

be a fuzzy location.

Because of $f'(x) \times 3x^2$, we obtain $f'(\tilde{X}_0) = \{(0, 1), (3, .6)\}$ as derivative of a real-valued function at the fuzzy point $\tilde{X}_0$.

Proposition 7–5

The extended sum $\oplus$ of the derivatives of two real-valued functions f and g at the fuzzy point $\tilde{X}_0$ is defined by

$$\mu_{(f'+g')(\tilde{x}_0)}(y) = \sup_{xy=f'(x)+g'(x)} \mu_{\tilde{x}_0}(x)$$

Hence

$$f'(\tilde{X}_0) \oplus g'(\tilde{X}_0) \supseteq (f' + g')\tilde{X}_0$$

Proposition 7–6 [Dubois and Prade 1982b, p. 227]

If f' and g' are continuous and both are nondecreasing or nonincreasing, then

$$f'(\tilde{X}_0) \oplus g'(\tilde{X}_0) = (f' + g')\tilde{X}_0$$

Proposition 7–7 (*Chain rule of differentiation*)

1. $(f \cdot g)'(\tilde{X}_0) = (f'g + fg')(\tilde{X}_0) \subseteq [f'(\tilde{X}_0) \odot g(\tilde{X}_0)] \oplus [f(\tilde{X}_0) \odot g'(\tilde{X}_0)]$
2. If f, g, f, and g' are continuous, f and g are both positive, and f' and g' are both nondecreasing (f, g is negative and f', g' is nondecreasing) then

$$(f \cdot g)'(\tilde{X}_0) = (f'(\tilde{X}_0) \odot g(\tilde{X}_0)] \oplus [f(\tilde{X}_0) \odot g'(\tilde{X}_0)]$$

Exercises

1. Determine the maximizing set of

$$f(x) = \begin{cases} 2x^2 - 3 & -2 \le x \le 2 \\ 5 & \text{else} \end{cases}$$

2. Show that computing $\mu \int_a^b b_f$ according to the extension principle yields the usual integral if $\tilde{f}$ is a crisp function.
3. Let $\tilde{f}(x) = (f(x), s(x), t(x))_{LR}$ with

$$f(x) = nx$$

$$s(x) = \frac{1}{|x|+1}$$

$$t(x) = \frac{1}{1+\sin^2 x}$$

$$L(x) = \frac{1}{1+2\,|x|^3}$$

$$L(x) = \frac{1}{1+x^{1/2}}$$

Determine $\tilde{f}(x)$ explicitly for $x = .5$, $x = 1$, and $x = 2$. Compute the integral $\tilde{I}(a, b)$.

4. Let $f(x) = 2x^3 + (x - 1)^2$,

$$\tilde{X}_0 = \{(-1, .5), (0, .8), (1, 1), (2, .6), (3, .4)\}$$

Computer $f'(\tilde{X}_0)$. Verify that proposition 7–6 holds.

5. Let $(\tilde{X}_0) = \{(-1, .4), (0, 1), (1, .6)\}$,

$$f(x) = x^3 + 2 \quad g(x) = 2x + 3$$

Compute $f'(\tilde{X}_0)$. Verify that proposition 7–6 holds.

8 UNCERTAINTY MODELING

8.1 Application-oriented Modeling of Uncertainty

As already mentioned in section 1.1, the type of uncertainty modeling chosen is entirely up to the modeler if and when a formal model is under consideration which does not pretend to model reality correctly.

If, however, the modeler is faced with a real application, then he still has a certain freedom of choice but he is also limited by the character of the piece of reality he wants to model.

The modeler of such a problem will have to decide whether he wants to consider uncertainty—defined in whatever way—explicitly in his model or not. He might, for instance, prefer to approximate the uncertain phenomenon by a certain (deterministic) model. Alternatively he might include as much "slack" in his model that he is "on the safe side" concerning uncertainty, or he might prefer a "wait and see" solution by waiting with a decision until in the pass of time uncertainty has almost disappeared. This would amount to reducing the influence of uncertainty by reducing its causes which, of course, have to be known in this case. In either of the above cases the modeler does not have to choose any specific method for modeling uncertainty. In the rest of the chapter we shall focus on those cases in which the modeler decides to model uncertainty explicitly.

Until the 1960s probability theories and statistics were the only methods to model uncertainty which has always been considered by scientists as a rather disturbing feature of some scientific statements, of systems, phenomena or even in philosophy. Since the 1960s additional theories have been suggested as tools to model uncertainty. Some of these theories or their supporters even claim to be the only proper tool for modeling uncertainty, even though the notion of uncertainty has never been defined uniquely.

It has been defined in *specific* contexts—mainly formal theories—but then the semantic interpretation is generally restricted to this field. In decision logic, for instance, "decisions under uncertainty" are defined as acts of choice for which the state of the nature that will occur is unknown. Unluckily, as Schneider already observed in 1979, those situations occur in practice very seldomly, if at all [Schneider 1979].

One would expect to find an appropriate definition of uncertainty either in lexica or in scholarly books on "uncertainty" modeling [Goodman and Nguyen 1985, Klir and Folger 1988, Klir 1987]. Surprisingly enough I have not been successful to find any general definition for it.

The first question one should probably ask is whether uncertainty is a phenomenon, a feature of real world systems, a state of mind or a label for a situation in which a human being wants to make statements about phenomena (i.e. reality, models, theories). One can also ask whether "uncertainty" is an objective fact or just a subjective impression which is closely related to individual persons.

Whether uncertainty is an objective feature of physical real systems seems to be a philosophical question. In the following we shall *not* consider these "objective uncertainties" if they exist, but we shall focus on the human-related, subjective interpretation of "uncertainty" which depends on the quantity and quality of information which is available to a human being about a system or its behavior that the human being wants to describe, predict or prescribe.

In this respect it shall not matter whether the information is inadequate due to the specific individuum or whether it is due to the present state of knowledge, i.e. whether the information is not available at present to anybody. Figure 8–1 depicts our view of uncertainty used in this chapter.

In this figure the "system" denotes the phenomenon about which judgments are to be made. This can be parts of the physical reality, socio-economic systems, man-made systems or any other type of phenomena. Information or data emitted by the system might be impulses, visible or measurable properties (noise, temperature etc.). Theses data or information are, however, very often not considered directly by the "observer". They are rather the input to an uncertainty theory (e.g. probability theory), which processes this information in specified ways and supplies the observer with certain "measures of uncertainty" (e.g. mean values,

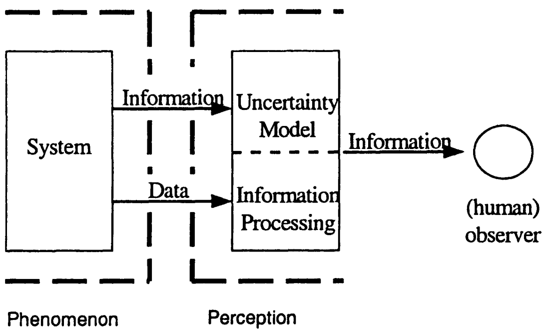


Figure 8–1. Uncertainty as situational property.

variances etc.) or descriptions of uncertainty (e.g. probability distributions etc.). Hence, the observer does not perceive the information about the phenomenon directly but only after it has been "filtered" by the uncertainty theory used.

The most important aspects of this view are:

1. "Causes" of uncertainty influence the information flow between the observed system and the uncertainty model (paradigm chosen by the observer).
2. A selected uncertainty model or theory has to be appropriate to the available quantity and quality of input information.
3. A chosen uncertainty theory also determines the type of information processing applied to available data or information.
4. For pragmatic reasons the information offered to the observer (human or other) by the uncertainty model should be in an adequate language.
5. Hence, the choice of an appropriate "uncertainty" calculus may depend on
   - the causes of uncertainty,
   - quantity and quality of information available,
   - type of information processing required by the respective "uncertainty" calculus and
   - language required by the final observer.

Even this notion of uncertainty is rather vague, has many different appearances and many different causes. It is, therefore, difficult to define it properly and in sufficient generality. Any definition of uncertainty is in a way arbitrary and subjective. It can be more or less extreme with respect to the situation. Here we chose a rather broad definition for uncertainty in order to include a large number of possible situations which can be considered "uncertain".

Definition 8–1: A proposed definition of uncertainty

Uncertainty implies that in a certain situation a person does not dispose about information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behavior or other characteristica.

"Situation" in the context of this definition includes features of the system as well as expectations or needs of the observer. The need to describe a phenomenon numerically was included because most of the known measures of uncertainty require a numerical description. In some situations a symbolic description of the phenomenon may be sufficient for the human observer to judge the situation (e.g. the color of the traffic lights at a road intersection). But in this case he knows in addition to the color the meaning of the color and he will not be in a position to make statements about the traffic behavior at an intersection without involving numbers.

It seems that a lot of misunderstandings have been caused by confusing the "type of uncertainty" with the "cause of uncertainty" or with the theory which is used to model uncertainty. I shall, therefore, attempt to describe in the following these three aspects of uncertainty separately in order to arrive at a certain taxonomy of uncertainty, the classes of which may neither be disjunct nor exhaustive.

8.1.1 Causes of Uncertainty

Lack of Information. Lack of information is probably the most frequent cause for uncertainty. In decision logic, for instance, one calls "decisions under uncertainty" the situation in which a decision maker does not have any information about which of the possible states of nature will occur. This would obviously be a quantitative lack of information. With "decision making under risk" one normally describes a situation in which the decision maker knows the probabilities for the occurrence of various states. This could be called a qualitative lack of information. Since information about the occurrence is available, it can also be considered complete in the sense of the availability of a complete probability function. But the kind of the available information is not sufficient to describe the situation deterministically. Another situation characterized by a lack of information might be called "approximation". Here one does not have or one does not want to gather sufficient information to make an exact description, even though this might be possible. In some cases the description of the system is explicitly called an "approximation", in other situations this is hidden and probably not visible to the normal observer. Examples for the latter case can be found in mathematics where symbols are used rather than real numbers because a description by real numbers is not feasible (for instance the "number" π, sin and cosine functions, or any complex or transcendental numbers). In this context the scale level on which numerical information is available also has to be considered. The situation of "certainty" normally assumes an absolute or at least a cardinal scale level of the information available. If only information on a ratio, ordinal or nominal scale level is available, this would also be called a "qualitative lack of information" in our view.

A transition from a situation of uncertainty caused by a lack of information to a situation of certainty can obviously only be achieved by gathering more or better information. Whether this is possible or desirable obviously depends on the situation and the goal of modeling.

Abundance of Information (Complexity). This type of uncertainty is due to the limited ability of human beings to perceive and process simultaneously large amounts of data [Newell and Simon 1972]. This situation is exemplified by real world situations in which more data is objectively available to human beings than they can "digest" or by situations in which human beings communicate about phenomena which are defined or described by a large number of features or properties. What people do in these situations is normally, that they transform the available data into perceivable information by using a coarser grid or a rougher "granularity" or by focusing their attention on those features which seem to them most important and neglecting all other information or data. If such a situation occurs in scientific activities, very often some kind of "scaling" is used to the same end. It is obvious that in these situations a transfer to "certainty" cannot be achieved by gathering even more data, but rather by transforming available data to appropriate information.

Conflicting Evidence. Uncertainty might also be due to conflicting evidence, i.e. there might be considerable information available pointing to a certain behavior of a system and additionally there might also be information available pointing to another behavior of the system. If the two classes of available information are conflicting, then an increase of information might not reduce uncertainty at all, but rather increase the conflict. The reason for this conflict of evidence can certainly be different. It can be due to the fact that some of the information available is wrong (but not identifiable as wrong information by the system), it can also be that information of non-relevant features of the system is being used, it might be that the model which the observer has of the system is wrong etc. In this case a transition to a situation of certainty might call for checking the available information again with respect to the correctness rather than gathering more information or putting the information on a rougher grid. In some cases, however, deleting some pieces of information might reduce the conflict and move the situation closer in the direction of certainty.

Ambiguity. By ambiguity we mean a situation in which certain linguistic information, for instance, has entirely different meanings or in which—mathematically speaking—we have a one-to-many mapping. All languages contain certain words which for several reasons have different meanings in different contexts. A human observer can normally easily interpret the word correctly semantically if he knows the context of the word. In so far this type of uncertainty could also be classified under "lack of information" because in this case adding more information about the context to the word may move us from uncertainty to certainty.

Measurement. The term "measurement" also has very different interpretations in different areas [Zimmermann and Zysno 1980]. In the context of this chapter we mean "measurement" in the sense of "engineering measurement", i.e. of measuring devices to measure physical features, such as weight, temperature, length etc.

The quality of our measuring technology has increased with time and the further this technology improves, the more exactly it can determine properties of physical systems. As long, however, as an "imagined" exact property cannot yet be measured perfectly, we have some uncertainty about the real measure and we only know the indicated measure. This is certainly also some type of uncertainty which could also be considered as a "lack of information". It is only considered to be a separate class in this paper due to the particular importance of this type of uncertainty to engineering.

Belief. Eventually, we would like to mention as cause of uncertainty situations in which all information available to the observer is subjective as a kind of belief in a certain situation. This situation is probably most disputable and it could also be considered as "lack of information" in the objective sense.

A possible interpretation of this situation is, however, also that a human being develops on the basis of available (objective) data and in a way which is unknown to us (subjective) beliefs which he afterwards considers as information about a system that he wants to describe or prescribe. The distinction of this class from the classes mentioned above is actually that, so far, we always have considered "objective" information and now we are moving to "subjective" information. Whether this distinction can and should be upheld at all is a matter for further discussion.

8.1.2 Type of Available Information

So far we have discussed causes of uncertainty which in most cases depend on the quality or quantity of available information. As already mentioned, however, we will have to consider the type of available information in a situation which we want to judge with respect to uncertainty in more detail: the information which is available for a system under consideration can, roughly speaking, be numerical, linguistic, interval-valued or symbolic.

Numerical Information. In our definition of certainty we requested that a system can be described numerically. This normally requires that the information about the system is also available numerically. Since this numerical information can come from quite a variety of sources, it is not sufficient to require just that the information is given in numbers, but we also have to determine the scale level on which this information is provided [Sneath and Sokal 1973]. This determines the type of information processing (mathematical operation) which we can apply to this information legitimately without pretending information which is not available. There is quite a number of taxonomies for scale levels, such as, for instance, distinguishing between nominal scale level, ordinal scale level, ratio scale level, interval scale level and absolute scale level. For our purposes we refer the reader to table 16–1.

Roughly speaking, a nominal scale level indicates that the information provided (even though in numerical form) only has the function of a name (such as the number on the back of a football player or a license plate of a car), that numerical information on an ordinal scale level provides information of an ordering type and information on a cardinal scale level also indicates information about the differences between the ordered quantities, i.e. contains a metric.

Interval-Information. In this case information is available, but not as precise in the sense of a real-valued number as above. If we want to process this information properly, we will have to use interval arithmetic and the outcome will again be interval-valued information. It should be clear, however, that this information is also "exact" or "dichotomous" in the sense that the boundaries of the intervals, no matter how they have been determined, are "crisp", "dichotomous", or "exact".

Linguistic Information. By linguistic information we mean that the information provided is given in a natural language and not in a formal language [Bellman and Zadeh 1970]. The properties of the type of information obviously differ from those of either numerical information or of information in a formal language. Natural languages develop over time, they depend on cultural backgrounds, they depend on educational backgrounds of the persons using this language and on many other things. One also has to distinguish between a word as a label and the meaning of a word. Very often there is neither a one-to-one relationship between these two nor are the meanings of words defined in a crisp and a context-independent way. By contrast to numerical information there are also hardly any measures of quality of information for natural languages (e.g. there are no defined scale levels for linguistic information). Linguistic information has developed as a means of communication between human beings and the "inference engines" are the minds of people about which is still much too little known.

Symbolic Information. Very often information is provided in the form of symbols. This is obvious when numbers, letters or pictures are being used as symbols. This is often not as obvious if words are being used as symbols because sometimes it seems to be suggested or assumed that words have natural meanings while symbols do not. Hence, if symbolic information is provided, the information is as valuable as the definitions of the symbols are and the type of information processing also has to be symbolic and neither numerical nor linguistic.

8.1.3 Uncertainty Methods

As depicted in figure 8–1, information of the uncertain phenomenon is filtered by an uncertainty method before it is offered to the observer. By "uncertainty methods" we mean any of the probability theories, fuzzy set theory, rough set theory, evidence theory etc. These theories build on certain axioms with respect to the uncertainty to be modeled and they propose generally a mathematical framework to arrive at measures of uncertainty [Dubois and Prade 1989]. The mathematical models or methods suggested require a certain scale level of numerical information. Hence, a specific uncertainty method should not be used if its mathematical operations require a higher scale level than that on which the available information is provided. This is very often neglected when applying those theories. Rather one assumes, without checking, that numerical information is available on a cardinal or absolute scale level for which all mathematical operations would be legitimate.

To an increasing degree, moreover, uncertain information or information about "uncertainties" is also processed in knowledge-based systems [Zimmermann 1988, Kandel and Langholz 1992, Klein and Methlie 1995, Turban 1988] which can either be systems which essentially perform symbol processing (classical expert system technology) or they perform meaning preserving inference. Obviously, for these systems different requirements exist and different types of information are offered at the end. Eventually, information can be processed heuristically, i.e. according to well-defined procedures which can also require other types of languages.

To model, i.e. describe, prescribe or predict, a system or the behavior of a system normally serves a certain purpose. It could serve a human observer, it could be the input to another mechanical or electronic system, it could be used for other mathematical algorithms etc. In figure 8–1 a human observer was considered as the recipient of the information. In this case the information does not only have to be "readable" by the recipient, but it may have to meet additional requirements, depending on what it is intended for. If the observer wants to recognize certain patterns, a nominal scale level of the received information might already be sufficient. If he wants to evaluate or order phenomena, information will have to be at least on an ordinal scale level, etc. Hence, the information about the uncertain system will have to be provided in a suitable language, i.e. either numerical, in the form of intervals, linguistically or symbolically, and on an appropriate scale level.

8.1.4 Uncertainty Theories as Transformers of Information

Sections 8.1.1 to 8.1.3 of this chapter focused on informational features of the uncertain phenomenon. The uncertainty calculus, theory or method used to describe this phenomenon should obviously be compatible with the features of the phenomenon, i.e. not require information on a higher level than provided, not make any axiomatic assumptions about the cause of uncertainty etc. which are not satisfied by the real situation.

This certainly contradicts views that, for instance, any uncertainty can be modeled by probabilities, or by fuzzy sets, or by possibilities, or by any other single method. We do not believe that there exists any single method which is able to model all types of uncertainty equally well.

Most of the established theories and methods for uncertainty modeling are focused either on specific "types of uncertainty" defined by their causes or they at least imply certain causes and they also require specific types or qualities of information depending on the type of information processing they use. One could consider these uncertainty methods and their paradigms as glasses through which we consider uncertain situations or with other words: there is no "probabilistic uncertainty" as distinct from "possibilistic uncertainty". One is rather looking at an uncertain situation with the properties that were specified before and one tries to model this uncertain situation by means of probability theory or by means of possibility theory. Hence, the theory which is appropriate to model a specific uncertainty situation should be determined by the properties of this situation as specified above and by the requirements of the observer. At present there exist numerous uncertainty theories, such as: various probability theories, evidence theory [Shafer 1976], possibility theory [Dubois and Prade 1988], fuzzy set theory, grey set theory, intuitonistic set theory [Atanassov 1986], rough set theory [Pawlak 1985], interval arithmetic, convex modeling [Ben-Haim and Elishakoff 1990], etc. Some of these theories are contained in other theories which shall not be investigated here.

We would like to point to one fact, however, which is sometimes overlooked: uncertainty theories are often not homogeneous with respect to their information processing or requirements as to the quality of information. Fuzzy set theory, for instance, claims to process linguistic information. The formal presentation of this information can be quite different. If singletons are used, this corresponds to symbol processing. If linguistic variables are used, the membership functions of the terms are processed. They can be on various scale levels and will, therefore, determine which operators, i.e. mathematical operations, may be used and which not.

Whether an uncertainty theory uses mathematical, heuristic or knowledge-based information processing or inference will also influence the type of required input information and the quality of the information offered to the observer.

8.1.5 Matching Uncertainty Theory and Uncertain Phenomena

Considering uncertainty as an informational feature of a situation or a phenomenon, it can be described by a 4-component vector. In this vector the four components describe the four dimensions which are roughly sketched in table 8–1.

Essentially each uncertainty theory can also be characterized by such a vector or profile. Optimally the profile of the theory should match the profile of the situation it is applied to.

For the most common frequentistic probability theory (Kolmogoroff) it is rather simple to define its profile, which is:

$$\{a;\ a;\ c;\ a\}.$$

In addition, some other properties, i.e. that the events have to be dichotomous etc., have to be assumed. For other probability theories it is already more difficult to determine an appropriate profile. For Fuzzy set theory the profile vector will certainly depend on the operators used, on the type of membership function assumed, on the scale level of the membership function etc. Or, putting it the

Table 8–1. Rough taxonomy of uncertainty properties.
Rough taxonomy of uncertainty models (not exhaustive, not disjunct).

1. *Causes of (subj.) uncertainty*
   (a) Lack of information
   (b) Abundance of information
   (c) Conflicting evidence
   (d) Ambiguity (complexity)
   (e) Measurement
   (f) Belief
2. *Available Information (Input)*
   (a) Numerical
   (b) Set- or interval-valued
   (c) Linguistic
   (d) Symbolic
3. *Scale Level of Numerical Information*
   (a) Nominal
   (b) Ordinal
   (c) Cardinal
4. *Required Information (Output)*
   (a) Numerical
   (b) Set- or interval-valued
   (c) Linguistic
   (d) Symbolic

other way around, after the "uncertainty profile" of the uncertain situation has been determined that version of fuzzy set theory that matches the profile of the situation has to be found.

In the following we shall compare to a certain degree three formal theories that have been developed either to model uncertainty (e.g. probability) or which are recommended amongst other goals for uncertainty modeling: probability theory, possibility theory and fuzzy set theory. We will also consider some "hybrid" notions, i.e. terms in which two (formal) theories have been combined. Since L. Zadeh proposed the concept of a fuzzy set in 1965, the relationships between probability theory and fuzzy set theory have been further discussed. Both theories seem to be similar in the sense that both are concerned with some type of uncertainty and both use the [0, 1] interval for their measures as the range of their respective functions (At least as long as one considers normalized fuzzy sets only!). Other uncertainty measures, which were already mentioned in chapter 4, also focus on uncertainty and could therefore be included in such a discussion. The comparison between probability theory and fuzzy set theory is difficult primarily for two reasons:

1. The comparison could be made on very different levels, that is, mathematically, semantically, linguistically, and so on.
2. Fuzzy set theory is not or is no longer a uniquely defined mathematical structure, such as Boolean algebra or dual logic. It is rather a very general family

> of theories (consider, for instance, all the possible operations defined in chapter 3 or the different types of membership functions). In this respect, fuzzy set theory could rather be compared with the different existing theories of multivalued logic.

Further, there does not yet exist and probably never will exist a unique context-independent definition of what fuzziness really means. On the other hand, neither is probability theory uniquely defined. There are different definitions and different linguistic appearances of "probability."

In recent years, some specific interpretations of fuzzy set theory have been suggested. One of them, possibility theory, used to correspond, roughly speaking, to the min-max version of fuzzy set theory—that is, to fuzzy set theory in which the intersection is modeled by the min-operator and the union by the max-operator. This interpretation of possibility theory, however, is no longer correct. Rather, it has been developed into a well-founded and comprehensive theory. After the basic articles by L. Zadeh [1978, 1981], most of the advances in possibility theory have been due to Dubois and Prade. See, for instance, their excellent book on this topic [Dubois and Prade 1988].

We shall first describe the essentials of possibility theory and then compare it with other theories of uncertainty.

8.2 Possibility Theory

8.2.1 Fuzzy Sets and Possibility Distributions

Possibility theory focuses primarily on imprecision, which is intrinsic in natural languages and is assumed to be "possibilistic" rather than probabilistic. Therefore the term *variable* is very often used in a more linguistic sense than in a strictly mathematical one. This is one reason why the terminology and the symbolism of possibility theory differ in some respects from those of fuzzy set theory. In order to facilitate the study of possibility theory, we will therefore use the common possibilistic terminology but will always show the correspondence to fuzzy set theory.

Suppose, for instance, we want to consider the proposition "X is $\tilde{F}$," where X is the name of an object, a variable, or a proposition. For instance, in "X is a small integer," X is the name of a variable. In "John is young," John is the name of an object. $\tilde{F}$ (i.e., "small integer" or "young") is a fuzzy set characterized by its membership function $\mu_{\tilde{F}}$.

One of the central concepts of possibility theory is that of a possibility distribution (as opposed to a probability distribution). In order to define a possibility distribution, it is convenient first to introduce the notion of a fuzzy restriction. To visualize a fuzzy restriction, the reader should imagine an elastic suitcase that acts on the possible volume of its contents as a constraint. For a hardcover suitcase, the volume is a crisp number. For a soft valise, the volume of its contents depends to a certain degree on the strength that is used to stretch it. The variable in this case would be the volume of the valise; the values this variable can assume may be $u \in U$, and the degree to which the variable (X) can assume different values of u is expressed by $\mu_{\tilde{F}}(u)$. Zadeh [Zadeh et al. 1975, p. 2; Zadeh 1978, p. 5] defines these relationships as follows.

Definition 8–2

Let $\tilde{F}$ be a fuzzy set of the universe U characterized by a membership function $\mu_{\tilde{F}}(u)$. $\tilde{F}$ is a *fuzzy restriction* on the variable X if $\tilde{F}$ acts as an elastic constraint on the values that may be assigned to X, in the sense that the assignment of the values u to X has the form

$$X = u \colon \mu_{\tilde{F}}(u)$$

$\mu_{\tilde{F}}(u)$ is the degree to which the constraint represented by $\tilde{F}$ is satisfied when u is assigned to X. Equivalently, this implies that $1 - \mu_{\tilde{F}}(u)$ is the degree to which the constraint has to be stretched in order to allow the assignment of the values u to the variable X.

Whether a fuzzy set can be considered as a fuzzy restriction or not obviously depends on its interpretation: This is only the case if it acts as a constraint on the values of a variable, which might take the form of a linguistic term or a classical variable.

Let $\tilde{R}(X)$ be a fuzzy restriction associated with X, as defined in definition 8–1. Then $\tilde{R}(X) = \tilde{F}$ is called a *relational assignment equation*, which assigns the fuzzy set F to the fuzzy restriction $\tilde{R}(X)$.

Let us now assume that $A(X)$ is an implied attribute of the variable X—for instance, $A(X)$ = "age of Jack," and $\tilde{F}$ is the fuzzy set "young." The proposition "Jack is young" (or better "the age of Jack is young") can then be expressed as

$$\tilde{R}(A(X)) = \tilde{F}$$

Example 8–1 [Zadeh 1978, p. 5]

Let p be the proposition "John is young," in which "young" is a fuzzy set of the universe $U = [0, 100]$ characterized by the membership function

$$\mu_{\text{young}}(u) = S(u;\ 20,\ 30,\ 40)$$

where u is the numerical age and the S-function is defined by

$$S(u;\alpha,\beta,\gamma)=\begin{cases}1 & \text{for} \quad u<\alpha \\ 1-2\left(\dfrac{u-\alpha}{\gamma-\alpha}\right)^2 & \text{for} \quad \alpha\le u\le\beta \\ 2\left(\dfrac{u-\gamma}{\gamma-\alpha}\right)^2 & \text{for} \quad \beta<u\le\gamma \\ 0 & \text{for} \quad u>\gamma\end{cases}$$

In this case, the implied attribute $A(X)$ is Age (John), and the translation of "John is young" has the form

$$\text{John is young} \rightarrow \tilde{R}(\text{Age(John)}) = \text{young}$$

Zadeh [1978] related the concept of a fuzzy restriction to that of a possibility distribution as follows:

> Consider a numerical age, say u = 28, whose grade of membership in the fuzzy set "young" is approximately 0.7. First we interpret 0.7 as the degree of compatibility of 28 with the concept labelled young. Then we postulate that the proposition "John is young" converts the meaning of 0.7 from the degree of compatibility of 28 with young to the degree of possibility that John is 28 given the proposition "John is young." In short, the compatibility of a value of u with young becomes converted into the possibility of that value of u given "John is young" [Zadeh 1978, p. 6].

The concept of a possibility distribution can now be defined as follows:

Definition 8–3 [Zadeh 1978, p. 6]

Let $\tilde{F}$ be a fuzzy set in a universe of discourse U that is characterized by its membership function $\mu_{\tilde{F}}(u)$, which is interpreted as the compatibility of $u \in U$ with the concept labeled $\tilde{F}$.

Let X be a variable taking values in U, and let $\tilde{F}$ act as a fuzzy restriction, $\tilde{R}(X)$, associated with X. Then the proposition "X is $\tilde{F}$," which translates into $\tilde{R}(X) = \tilde{F}$, associates a *possibility distribution*, π_x, with X that is postulated to be equal to $\tilde{R}(X)$.

The possibility distribution function, $\pi_x(u)$, characterizing the possibility distribution π_x is defined to be numerically equal to the membership function $\mu_{\tilde{F}}(u)$ of $\tilde{F}$, that is,

$$\pi_x \triangleq \mu_{\tilde{F}}$$

The symbol $\triangleq$ will always stand for "denotes" or "is defined to be." In order to stay in line with the common symbol of possibility theory, we will denote a possibility distribution with π_x rather than with $\tilde{\pi}_x$, even though it is a fuzzy set.

Example 8–2 [Zadeh 1978, p. 7]

Let U be the universe of positive integers, and let $\tilde{F}$ be the fuzzy set of small integers defined by

$$\tilde{F} = \{(1,1),(2,1),(3,.8),(4,.6),(5,.4),(6,.2)\}$$

Then the proposition "X is a small integer" associates with X the possibility distribution

$$\pi_x = \tilde{F}$$

in which a term such as (3, .8) signifies that the possibility that X is 3, given that X is small integer, is .8.

Even though definition 8–3 does not assert that our intuition of what we mean by possibility agrees with the min-max fuzzy set theory, it might help to realize their common origin. It might also make more obvious the difference between possibility distribution and probability distribution.

Zadeh [1978, p. 8] illustrates this difference by a simple but impressive example.

Example 8–3

Consider the statement "Hans ate X eggs for breakfast," $X = \{1, 2, \ldots\}$. A possibility distribution as well as a probability distribution may be associated with X. The possibility distribution $\pi_x(u)$ can be interpreted as the degree of ease with which Hans can eat u eggs while the probability distribution might have been determined by observing Hans at breakfast for 100 days. The values of $\pi_x(u)$ and $P_x(u)$ might be as shown in the following table:

u	1	2	3	4	5	6	7	8
$\pi_x(u)$	1	1	1	1	.8	.6	.4	.2
$P_x(u)$	.1	.8	.1	0	0	0	0	0

We observe that a high degree of possibility does not imply a high degree of probability. If, however, an event is not possible, it is also improbable. Thus, in a way, the possibility is an upper bound for the probability. A more detailed discussion of this "*possibility/probability consistency principle*" can be found in Zadeh [1978].

This principle is not intended as a crisp principle, from which exact probabilities or possibilities can be computed, but rather as a heuristic principle, expressing the principle relationship between possibilities and probabilities.

8.2.2 Possibility and Necessity Measures

In chapter 4, a possibility measure was already defined (definition 4–2) for the case in which A is a crisp set. If $\tilde{A}$ is a fuzzy set, a more general definition of a possibility measure has to be given [Zadeh 1978, p. 9].

Definition 8–4

Let $\tilde{A}$ be a fuzzy set in the universe U, and let π_x be a possibility distribution associated with a variable X that takes values in U. The *possibility measure*, $\pi_x(\tilde{A})$, of $\tilde{A}$ is then defined by

$$\begin{aligned}\text{poss}\{X \text{ is } \tilde{A}\} &\triangleq \pi(\tilde{A}) \\ &\triangleq \sup_{u \in U} \min\{\mu_{\tilde{A}}(u), \pi_x(u)\}\end{aligned}$$

Example 8–4 [Zadeh 1978]

Let us consider the possibility distribution induced by the proposition "X is a small integer" (see example 8–2):

$$\pi_x = \{(1,1),(2,1),(3,.8),(4,.6),(5,.4),(6,.2)\}$$

and the crisp set $A = \{3, 4, 5\}$.

The possibility measure $\pi(A)$ is then

$$\pi(A) = \max(.8,.6,.4) = .8$$

If $\tilde{A}$, on the other hand, is assumed to be the fuzzy set "integers which are not small," defined as

$$\tilde{A} = \{(3,.2),(4,.4),(5,.6),(6,.8),(7,1),\ldots\}$$

then the possibility measure of "X is not a small integer" is

$$\text{poss}(X \text{ is not a small integer}) = \max\{.2, .4, .4, .2\} = .4$$

Similar to probability theory, conditional possibilities also exist. Such a conditional possibility distribution can be defined as follows [Zadeh 1981b, p. 81].

Definition 8–5

Let X and Y be variables in the universes U and V, respectively. The *conditional possibility distribution* of X given Y is then induced by a proposition of the form "If X is $\tilde{F}$, then Y is $\tilde{G}$" and is denoted by $\pi_{(Y/X)}(v/u)$.

Proposition 8–1

Let $\pi_{(Y/X)}$ be the conditional possibility distribution functions of X and Y, respectively. The joint possibility distribution function of X and Y, $\pi_{(X,Y)}$, is then given by

$$\pi_{(X,Y)}(u, v) = \min\{\pi_X(u), \pi_{(Y/X)}(v/u)\}$$

Not quite settled yet seems to be the question of how to derive the conditional possibility distribution functions from the joint possibility distribution function. Different views on this question are presented by Zadeh [1981b, p. 82], Hisdal [1978], and Nguyen [1978].

Fuzzy measures as defined in definition 4–2 express the degree to which a certain subset of a universe, Ω, or an event is possible. Hence, we have

$$g(0) = 0 \quad \text{and} \quad g(\Omega) = 1$$

As a consequence of condition 2 of definition 4–2, that is,

$$A \subseteq B \Rightarrow g(A) \leq g(B)$$

we have

$$g(A \cup B) \geq \max(g(A), g(B)) \quad \text{and} \tag{8.1}$$

$$g(A \cap B) \geq \min(g(A), g(B)) \quad \text{for} \quad A, B \subseteq \Omega \tag{8.2}$$

Possibility measures (definition 4–2) are defined for the limiting cases:

$$\pi(A \cup B) = \max\,(\pi(A), \pi(B)) \tag{8.3}$$

$$\pi(A \cap B) = \min\,(\pi(A), \pi(B)) \tag{8.4}$$

Table 8–2. Possibility functions.

Student	Grade				
	A	*B*	*C*	*D*	*E*
1	.8	1	.7	0	0
2	1	.8	.6	.1	0
3	.6	.7	.9	.1	0
4	0	.8	.9	.5	0
5	0	0	.3	1	.2
6	.3	1	.3	0	0

If $\complement A$ is the complement of A in Ω, then

$$\pi(A \cup \complement A) = \max(\pi(A), \pi(\complement A)) = 1 \tag{8.5}$$

which expresses the fact that either A or $\complement A$ is completely possible.

In possibility theory, an additional measure is defined that uses the conjunctive relationship and, in a sense, is dual to the possibility measure:

$$N(A \cap B) = \min(N(A), N(B)) \tag{8.6}$$

N is called then necessity measure. $N(A) = 1$ indicates that A is necessarily true (A is sure). The dual relationship of possibility and necessity requires that

$$\pi(A) = 1 - N(\complement A);\ \forall A \subseteq \Omega \tag{8.7}$$

Necessity measures satisfy the condition

$$\min(N(A), N(\complement A)) = 0 \tag{8.8}$$

The relationships between possibility measures and necessity measures satisfy the following conditions [Dobois and Prade 1988, p. 10]:

$$\pi(A) \geq N(A), \quad \forall A \subseteq \Omega \tag{8.9}$$

$$N(A) > 0 \Rightarrow \pi(A) = 1$$

$$\pi(A) < 1 \Rightarrow N(A) = 0 \tag{8.10}$$

Here Ω is always assumed to be finite.

Example 8–5

Let us assume that we know, from past experience, the performance of six students in written examinations. Table 8–1 exhibits the possibility functions for the grades A through E and students 1 through 6.

First we observe that the membership function for the grades of student 4 is not a possibility function, since $g(\Omega) \neq 1$.

We can now ask different questions:

1. How reliable is the statement of student 1 that he will obtain a B in his next exam?
   In this case, "A" is $\{B\}$ and "$\complement A$" is $\{A, C, D, E\}$.
   Hence, $\pi(A) = 1$

$$\begin{aligned} N(A) &= \min\{1 - \pi_i\} \\ &= \min\{.2, .3, 1, 1\} = .2. \end{aligned}$$

   Hence, the possibility of student 1 getting a B is $\pi = 1$, the necessity $N = .2$.
2. If we want to know the truth of the statement "Either student 1 or 2 will achieve an A or a B," our Ω has to be defined differently. It now contains the elements of the first two rows. The result would be

$$\pi(A) = \pi(\text{student 1 A or B or Student 2 A or B}) = 1$$

$$N(A) = .3$$

3. Let us finally determine the credibility of the statement "student 1 will get a C." In this case

$$\pi(A) = .7$$
$$N(A) = 0.$$

8.3 Probability of Fuzzy Events

By now it should have become clear that possibility is not a substitute for probability, but rather another kind of uncertainty.

Let us now assume that an event is not crisply defined except by a possibility distribution (a fuzzy set) and that we are in a classical situation of stochastic uncertainty, that is, that the happening of this (fuzzily described) event is not certain and that we want to express the probability of its occurence. Two views on this probability can be adopted: Either this probability should be a scalar (measure) or this probability can be considered as a fuzzy set also. We shall consider both views briefly.

8.3.1 Probability of a Fuzzy Event as a Scalar

In classical probability theory, an event A is a member of an α-field a of subsets of a sample space Ω. A probability measure P is a normalized measure over a measurable space (Ω, a)—that is, P is a real-valued function that assigns to every A in a a probability $P(A)$ such that

1. $P(A) \geq 0 \; A \in a$
2. $P(\Omega) = 1$
3. If $A_i \in a$, $i \in I \subset \mathbb{N}$, pairwise disjoint, then

$$P\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} P(A_i)$$

If Ω is, for instance, a Euclidean n-space and a the σ-field of Borel sets in $\mathbb{R}^n$, then the probability of A can be expressed as

$$P(A) = \int_A dP$$

If $\mu_A(x)$ denotes the characteristic function of a crisp set of A and $E_p(\mu_A)$ the expectation of $\mu_A(x)$, then

$$P(A) = \int_{R^n}^{\mu_A} (x)\, dP = E_P(\mu_A)$$

If $\mu_A(x)$ does not denote the characteristic function of a crisp set but rather the membership function of a fuzzy set, the basic definition of the probability of A should not change. Zadeh [1968] therefore defined the probability of a fuzzy event $\tilde{A}$ (i.e., a fuzzy set $\tilde{A}$ with membership function $\mu_{\tilde{A}}(x)$) as follows.

Definition 8–6

Let ($\mathbb{R}^n$, a, P) be a probability space in which a is the σ-field of Borel sets in $\mathbb{R}^n$ and P is a probability measure over $\mathbb{R}^n$. Then a *fuzzy event* in $\mathbb{R}^n$ is a fuzzy set $\tilde{A}$ in $\mathbb{R}^n$ whose membership function $\mu_{\tilde{A}}(x)$ is Borel measurable.

The *probability of a fuzzy event* $\tilde{A}$ is then defined by the Lebesque-Stieltjes integral

$$P(\tilde{A}) = \int_{R^n}^{\mu_A} (x)\, dP = E(\mu_{\tilde{A}})$$

In Zadeh [1968] the similarity of the probability of fuzzy events and the probability of crisp events is illustrated. His suggestions, though very plausible, were not yet axiomatically justified in 1968. Smets [1982] showed, however, that an axiomatic justification can be given for the case of crisp probabilities of fuzzy events within nonfuzzy environments. Other authors consider other cases, such as fuzzy probabilities, which we will not investigate in this book.

We shall rather turn to the definition of the probability of a fuzzy event as a fuzzy set, which corresponds quite well to some approaches we have discussed, for example, for fuzzy integrals.

8.3.2 Probability of a Fuzzy Event as a Fuzzy Set

In the following we shall consider sets with a finite number of elements. Let us assume that there exists a probability measure P defined on the set of all crisp subsets of (the universe) X, the Borel set. $P(x_i)$ shall denote the probability of element $x_i \in X$.

Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)|x \in X\}$ be a fuzzy set representing a fuzzy event. The degree of membership of element $x_i \in \tilde{A}$ is denoted by $\mu_{\tilde{A}}(x_i)$. α-level sets or α-cuts as already defined in definition 2–3 shall be denoted by A_α.

Yager [1979, 1984] suggests that it is quite natural to define the probability of an α-level set as $P(A_\alpha) = \Sigma_{x\in A_\alpha} P(x)$. On the basis of this, the probability of a fuzzy event is defined as follows [Yager 1984].

Definition 8–7

Let A_α be the α-level set of a fuzzy set $\tilde{A}$ representing a fuzzy event. Then the *probability of fuzzy event* $\tilde{A}$ can be defined as

$$P_Y(\tilde{A}) = \{(P(A_\alpha), \alpha)|\alpha \in [0, 1]\}$$

with the interpretation "the probability of at least an α degree of satisfaction to the condition $\tilde{A}$."

The subscript Y of P_Y indicates that P_Y is a definition of probability due to Yager that differs from Zadeh's definition, which is denoted by P. It should be very clear that Yager considers α, which is used as the degree of membership of the probabilities $P(A_\alpha)$ in the fuzzy set $P_Y(\tilde{A})$, as a kind of significance level for the probability of a fuzzy event.

On the basis of private communication with Klement, Yager also suggests another definition for the probability of a fuzzy event, which is derived as follows.

Definition 8–8

The *truth of the proposition* "the probability $\tilde{A}$ is at least w" is defined as the fuzzy set $P_y^*(\tilde{A})$ with the membership function

$$P_y^*(\tilde{A})(w) = \sup_\alpha\{\alpha|P(A_\alpha) \geq w\}, \quad w \in [0, 1]$$

The reader should realize that now the "indicator" of significance of the probability measure is w and no longer α! The reader should also be aware of the fact that we have used Yager's terminology denoting the values of the membership function by $P_y^*(\tilde{A})(w)$. This will facilitate reading Yager's work [1984].

If we denote the complement of $\tilde{A}$ by $\complement\tilde{A} = \{(x, 1 - \mu_{\tilde{A}}(x))|x \in X\}$ and the α-level sets of $\complement\tilde{A}$ by $(\complement\tilde{A})_\alpha$, then $P_y^*(\complement A)(w) = \sup_\alpha \{\alpha|P(\complement\tilde{A})_\alpha \geq w\}$, and $w \in [0, 1]$ can be interpreted as the truth of the proposition "the probability of not $\tilde{A}$ is at least w."

Let us define $\bar{P}_y^*(\tilde{A}) = 1 - P_y^*(\complement\tilde{A})$. If $\bar{P}_y^*(\tilde{A})(w)$ is interpreted as the truth of the proposition "probability of $\tilde{A}$ is at most w," then we can argue as follows: The "and" combination of "the probability of $\tilde{A}$ is at least w" and "the probability of $\tilde{A}$ is at most w" might be considered as "the probability of $\tilde{A}$ is exactly w." If $P_y^*(\tilde{A})$ and $\bar{P}_y^*(\tilde{A})$ are considered as possibility distributions, then their conjunction is their intersection (modeled by applying the min-operator to the respective membership functions). Hence the following definition [Yager 1984]:

Definition 8–9 [Yager 1984]

Let $P_y^*(\tilde{A})$ and $\bar{P}_y^*(\tilde{A})$ be defined as above. The possibility distribution associated with the proposition "the probability of $\tilde{A}$ is exactly w" can be defined as

$$\bar{P}_y(\tilde{A})(w) = \min\left\{P_y^*(\tilde{A})(w), \bar{P}_y^*(\tilde{A})(w)\right\}$$

Example 8–6

Let $\tilde{A} = \{(x_1, 1), (x_2, .7), (x_3, .6), (x_4, .2)\}$ be a fuzzy event with the probability defined for the generic elements: $P_1 = .1$, $P_2 = .4$, $P_3 = .3$, and $P_4 = .2$; $p\{x_2\}$ is .4, where the element x_2 belongs to the fuzzy event $\tilde{A}$ with a degree of .7.

First we compute $P_y^*(\tilde{A})$. We start by determining the α-level sets A_α for all $\alpha \in [0, 1]$. Then we compute the probability of the crisp events A_α and give the intervals of w for which $P(A_\alpha) \geq w$. We finally obtain $P_y^*(\tilde{A})$ as the respective supremum of α.

The computing is summarized in the following table:

α	A_α	$P(A)$	w	$P_y^*(\tilde{A}) = \sup\alpha$
[0, .2]	$\{x_1, x_2, x_3, x_4\}$	1	[.8, 1]	.2
[.2, .6]	$\{x_1, x_2, x_3\}$	.8	[.5, .8]	.6
[.6, .7]	$\{x_1, x_2\}$	.5	[.1, .5]	.7
[.7, 1]	$\{x_1\}$	.1	[0, .1]	1

Analogously, we obtain for $\overline{P}_y^*(\tilde{A}) = 1 - P_y^*(\complement\tilde{A})$,

	$(\complement\tilde{A})_\alpha$	$P(\complement\tilde{A})_\alpha$	w	$\overline{P}_y^*(\complement\tilde{A})$	$\overline{P}_y(\tilde{A}) = 1 - P_y(\complement\tilde{A})$
0	$\{x_1, x_2, x_3, x_4\}$	1	[.9, 1]	0	.1
[0, .3]	$\{x_2, x_3, x_4\}$	.9	[.5, .9]	.3	.7
[.3, .4]	$\{x_3, x_4\}$	.5	[.2, .5]	.4	.6
[.4, .8]	$\{x_4\}$	.2	[0, .2]	.8	.2
[.8, 1]	0	0	0	1	0

The probability $\overline{P}_y(\tilde{A})$ of the fuzzy even $\tilde{A}$ is now determined by the intersection of the fuzzy sets $P_y^*(\tilde{A})$ and $\overline{P}_y^*(\tilde{A})$ modeled by the min-operator as in definition 8–9:

$$\overline{P}_y(\tilde{A})(w) = \begin{cases} 0, & w = 0 \\ .2, & w \in [0, .2] \\ .6, & w \in [.2, .8] \\ .2, & w \in [.8, 1] \end{cases}$$

Figure 8–2 illustrates the fuzzy sets $P_y^*(\tilde{A})(w)$, $\overline{P}_y^*(\tilde{A})$ and $\overline{P}_y(\tilde{A})(w)$.

8.4 Possibility vs. Probability

Questions concerning the relationship between fuzzy set theory and probability theory are very frequently raised, particularly by "newcomers" to the area of fuzzy sets. There are probably two major reasons for this. On the one hand, there are certain formal similarities between fuzzy set theory (in particular when using normalized fuzzy sets) and probability theory; on the other hand, in the past probabilities have been the only means for expressing "uncertainty." It seems appropriate and helpful, therefore, to shed some more light on this question.

In the introduction to this chapter, it was already mentioned that such a comparison is difficult because of the lack of unique definitions of fuzzy sets. This lack of a unique definition is due in part to the variety of suggested possibilities for mathematically defining fuzzy sets as well as operations on them, as indicated in chapters 2 and 3. It is also due to the many different kinds of fuzziness that can be modeled with fuzzy sets, as described in chapter 1.

Another problem is the selection of the aspects with respect to which these theories shall be compared (see the introduction to this chapter!).

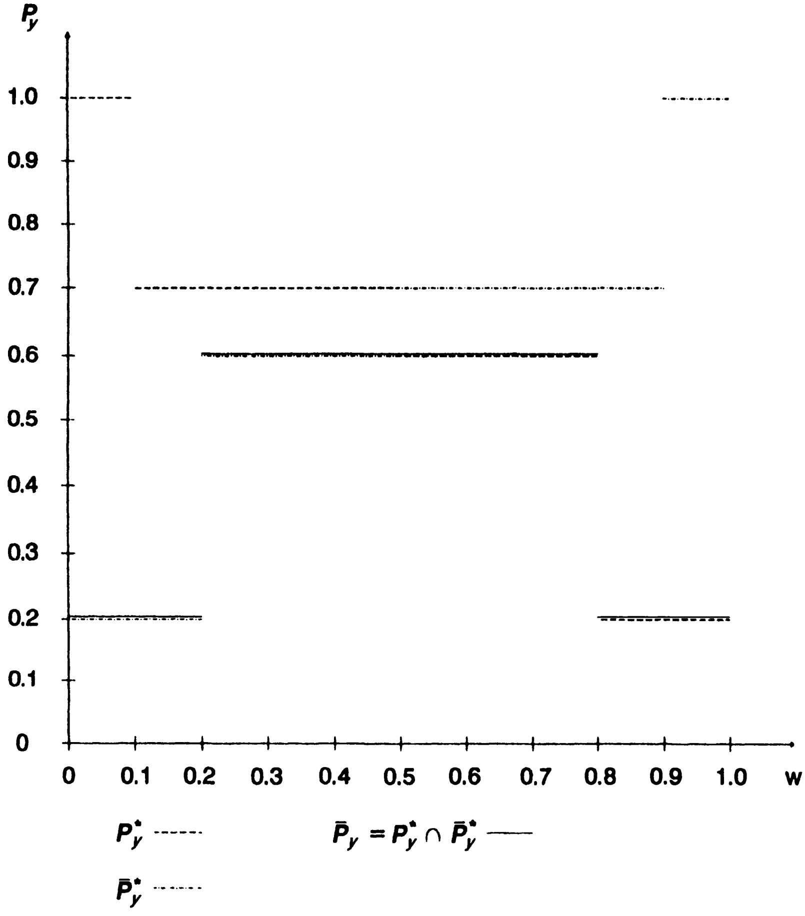


Figure 8–2. Probability of a fuzzy event.

In section 8.2, possibility theory was briefly explained. There it was mentioned that possibility theory is more than the min-max version of fuzzy set theory. It was also shown that the "uncertainty measures" used in possibility theory are the possibility measure and the necessity measure, two measures that in a certain sense are dual to each other. In comparing possibility theory with probability theory, we shall first consider only possibility functions—and measures (neglecting the existence of dual measures)—of possibility theory. At the end of the chapter, we shall investigate the relationship between possibility theory and probability theory.

Let us now turn to probabilities and try to characterize and classify available notions of probabilities. Three aspects shall be of main concern:

1. The linguistic expression of probability.
2. The different information context of different types of probabilities.
3. The semantic interpretation of probabilities and its axiomatic and mathematical consequences.

Linguistically, we can distinguish explicit from implicit formulations of probability. With respect to the information content, we can distinguish between probabilities that are classificatory (given E, H is probable), comparative (given E, H is more probable than K), partial (given E, the probability of K is in the interval $[a, b]$), and quantitative (given E, the probability of H is b).

Finally, the interpretation of a probability can vary considerably. Let us consider two very important and common interpretations of quantitative probabilities. Koopman [1940, pp. 269–292] and Carnap and Stegmüller [1959] interpret (subjective) probabilities essentially as degrees of truth of statements in dual logic. Axiomatically, Koopman derives a concept of probability, q, which mathematically is a Boolean ring.

Kolmogoroff [1950] interprets probabilities "statistically." He considers a set Ω and an associated σ-algebra $\mathcal{F}$, the elements of which are interpreted as events. On the basis of measurement theory, he defines a (probability) function P: $\mathcal{F} \to [0, 1]$ with the following properties:

$$P: I \to [0,1] \tag{8.11}$$

$$P(\Omega) = 1 \tag{8.12}$$

$$\forall (X_i) \in \mathcal{F} (\forall i, j \in \mathbb{N}: i \neq j \to X_i \cap X_j = \emptyset)\ P\Big(\bigcup_{i \in N} X_i\Big) = \sum_{i \in N} P(X_i) \tag{8.13}$$

From these properties, the following relationships can easily be derived:

$$X, \complement X \in \mathcal{F} \to P(\complement X) = 1 - P(X) \tag{8.14}$$

$$X, Y \in \mathcal{F} \to P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) \tag{8.15}$$

where $\complement X$ denotes the complement of X.

Table 8–3 illustrates the difference between Koopman's and Kolmogoroff's concept of probability, taking into account the different linguistic and informational possibilities mentioned above.

Now we are ready to compare "fuzzy sets" with "probabilities," or at least one certain version of fuzzy set theory with one of probability theory. Implicit probabilities re not comparable to fuzzy sets, since fuzzy set models try particularly

Table 8–3. Koopman's vs. Kolmogoroff's probabilities.

Koopman	*Kolmogoroff*
D, D', H, H' are statements of dual logic, Q is a nonnegative real number (generally $Q \in [0, 1]$)	W is a set of events, W_1 are subsets of W.
Classificatory:	
1. Implicit: D supports H	1. W_1 is a nonempty subset of W
2. Explicit: H is probable on the basis of D	2. If one throws the dice W times, probably no W_1 is empty.
Comparative:	
1. Implicit: D supports H more than D' supports H'	1. For W times one throws the dice, W_1 is of equal size as W_j.
2. H is more probable given D than H' is, given D'.	2. If one throws a coin W times, W_1 is as probable as W_j.
Quantitative:	
1. The degree of support for H on the basis of D is G.	1. The ratio of the number of events in W_1 and W is Q.
2. The probability for H given D is Q.	2. The probability that the result of throwing a dice is I when throwing the dice M times is Q_1.

to model uncertainty explicitly. Comparative and partial probabilities are more comparable to probabilistic statements using "linguistic variables," which we will cover in chapter 9.

Hence, the most frequently used versions we shall compare now are quantitative, explicit Kolmogoroff probabilities with possibilities.

Table 8–4 depicts some of the main mathematical differences between three areas that are similar in many respects.

Let us now return to the "duality" aspect of possibility measures and necessity measures.

A probability measure, $P(A)$, satisfies the additivity axiom, that is, $\forall A, B \subseteq \Omega$ for which $A \cap B = \emptyset$:

$$P(A \cup B) = P(A) + P(B) \tag{8.16}$$

This measure is monotonic in the sense of condition 2 of definition 4–2. Equation (8.12) is the probabilistic equivalent to (8.1) and (8.2).

The possibility theory conditions (8.5) and (8.8) imply

$$N(A) + N(\complement A) \leq 1 \tag{8.17}$$

Table 8–4. Relationship between Boolean algebra, probabilities, and possibilities.

	Boolean algebra	*Probabilities (quantitative explicit)*	*Possibilities*
Domain	Set of (logic) statements	σ-algebra	Any universe X
Range of values membership	$\{0, 1\}$	$[0, 1]$	$[0, 1]$ fuzzy: $0 < \mu < \infty$ real
Special constraints		$\sum_{\Omega} p(u) = 1$	
Union (independent, noninteractive)	max	Σ	max
Intersection	min	Π	min
Conditional equal to joint?	yes	no	often
What can be used for inference?	conditional	conditional or joint	conditional, often joint

$$\pi(A) + \pi(\complement A) \geq 1 \tag{8.18}$$

which is less stringent than the equivalent relation

$$P(A) + P(\complement A) = 1 \tag{8.19}$$

of probability theory.

In this sense, possibility corresponds more to evidence theory [Shafer 1976] than to classical probability theory, in which the probabilities of an element (a subset) are uniquely related to the probability of the contrary element (complement). In Shafer's theory, which is probabilistic in nature, this relationship is also relaxed by introducing an "upper probability" and a "lower probability," which are as "dual" to each other as are possibility and necessity.

In fact, possibility and necessity measures can be considered as limiting cases of probability measures in the sense of Shafer, that is,

$$N(A) \leq P(A) \leq \pi(A) \quad \forall A \subseteq \Omega \tag{8.20}$$

This in turn links intuitively again with Zadeh's "possibility/probability consistency principle" mentioned in section 8.2.1.

Concerning the theories considered in this chapter, we can conclude the following. Fuzzy set theory, possibility theory, and probability theory are no substitutes, but they complement each other. While fuzzy set theory has quite a number of "degrees of freedom" with respect to intersection and union operators, kinds of fuzzy sets (membership functions), etc., the latter two theories are well developed and uniquely defined with respect to operation and structure. Fuzzy set theory seems to be more adaptable to different contexts. This, of course, also implies the need to adapt the theory to a context if one wants it to be an appropriate modeling tool.

Exercises

1. Let U and $\tilde{F}$ be defined as in example 8–2. Determine the possibility distribution associated with the statement "X is not a small integer."
2. Define a probability distribution and a possibility distribution that could be associated with the proposition "cars drive X mph on American freeways."
3. Computer the possibility measures (definition 8–4) for the following possibility distributions:

   $$A = \{6, 7, \ldots, 13, 14\}$$

   "X is an integer close to 10"

   $$\pi_{\tilde{A}} = \{(8, .6), (9, .8), (10, 1), (11, .8), (12, .6)\}$$

   or alternatively,

   $$\pi_{\tilde{B}} = \{(6, .4), (7, .5), (8, .6), (9, .8), (10, 1), (11, .8), (12, .6), (13, .5), (14, .4)\}$$

   Discuss the results.
4. Discuss the relationships between general measures, fuzzy measures, probability measures, and possibility measures.
5. Determine Yager's probability of a fuzzy event for the event "X is an integer close to 10" as defined in exercise 3 above.
6. List examples for each of the kinds of probabilistic statements given in table 8–3.
7. Analyze and discuss the assertion that $\overline{P}_y^*(\tilde{A})(w)$ can be interpreted as the truth of the proposition "the probability of $\tilde{A}$ is at most w."

II APPLICATIONS OF FUZZY SET THEORY

Applications of fuzzy set theory can already be found in many different areas. One could probably classify those applications as follows:

1. Applications to mathematics, that is, generalizations of traditional mathematics such as topology, graph theory, algebra, logic, and so on.
2. Applications to algorithms such as clustering methods, control algorithms, mathematical programming, and so on.
3. Applications to standard models such as "the transportation model," "inventory control models," "maintenance models," and so on.
4. Finally, applications to real-world problems of different kinds.

In this book, the first type of "applications" will be covered by looking at fuzzy logic and approximate reasoning. The second type of applications will be illustrated by considering fuzzy clustering, fuzzy linear programming, and fuzzy dynamic programming. The third type will be covered by looking at fuzzy versions of standard operations research models and at multicriteria approaches. The fourth type, eventually, will be illustrated on the one hand by describing operations research (OR) models as well as empirical research in chapter 15. On the other hand, chapter 10 has entirely been devoted to fuzzy control and expert systems, the area in which fuzzy set theory has probably been applied to the largest extent and also which is closest to real applications.

9 FUZZY LOGIC AND APPROXIMATE REASONING

9.1 Linguistic Variables

> In retreating from precision in the face of overpowering complexity, it is natural to explore the use of what might be called *linguistic* variables, that is, variables whose values are not numbers but words or sentences in a natural or artificial language.
>
> The motivation for the use of words or sentences rather than numbers is that linguistic characterizations are, in general, less specific than numerical ones [Zadeh 1973a, p. 3].

This quotation presents in a nutshell the motivation and justification for fuzzy logic and approximate reasoning. Another quotation might be added, which is much older. The philosopher B. Russell noted:

> All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence [Russell 1923].

One of the basic tools for fuzzy logic and approximate reasoning is the notion of a linguistic variable that in 1973 was called a *variable of higher order* rather than a fuzzy variable and defined as follows [Zadeh 1973a, p. 75].

H.-J. Zimmermann, *Fuzzy Set Theory - and Its Applications*

Definition 9–1

A *linguistic variable* is characterized by a quintuple $(x, T(x), U, G, \tilde{M})$ in which x is the name of the variable; $T(x)$ (or simply T) denotes the term set of x, that is, the set of names of *linguistic values* of x, with each value being a fuzzy variable denoted generically by X and ranging over a universe of discourse U that is associated with the base variable u; G is a syntactic rule (which usually has the form of a grammar) for generating the name, X, of values of x; and M is a *semantic rule* for associating with each X its meaning, $\tilde{M}(X)$, which is a fuzzy subset of U. A particular X—that is, a name generated by G—is called a *term*. It should be noted that the base variable u can also be vector valued.

In order to facilitate the symbolism in what follows, some symbols will have two meanings wherever clarity allows this: x will denote the name of the variable ("the label") and the generic name of its values. The same will be true for X and $\tilde{M}(X)$.

Example 9–1 [Zadeh 1973a, p. 77]

Let X be a linguistic variable with the label "Age" (i.e., the label of this variable is "Age," and the values of it will also be called "Age") with $U = [0, 100]$. Terms of this linguistic variable, which are again fuzzy sets, could be called "old," "young," "very old," and so on. The base-variable u is the age in years of life. $\tilde{M}(X)$ is the rule that assigns a meaning, that is, a fuzzy set, to the terms:

$$\tilde{M}(\text{old}) = \{(u, \mu_{\text{old}}(u)) | u \in [0, 100]\}$$

where

$$\mu_{\text{old}}(u) = \begin{cases} 0 & u \in [0, 50] \\ \left(1 + \left(\dfrac{u-50}{5}\right)^{-2}\right)^{-1} & u \in (50, 100] \end{cases}$$

$T(x)$ will define the term set of the variable x, for instance, in the case

$T(\text{Age}) = \{$old, very old, not so old, more or less young, quite young, very young$\}$

where $G(x)$ is a rule that generates the (labels of) terms in the term set.

Figure 9–1 sketches another way to represent the linguistic variable "age".

Two linguistic variables of particular interest in fuzzy logic and in (fuzzy) probability theory are the two linguistic variables "Truth" and "Probability." The linguistic variable "Probability" is depicted exemplarily in figure 9–2.

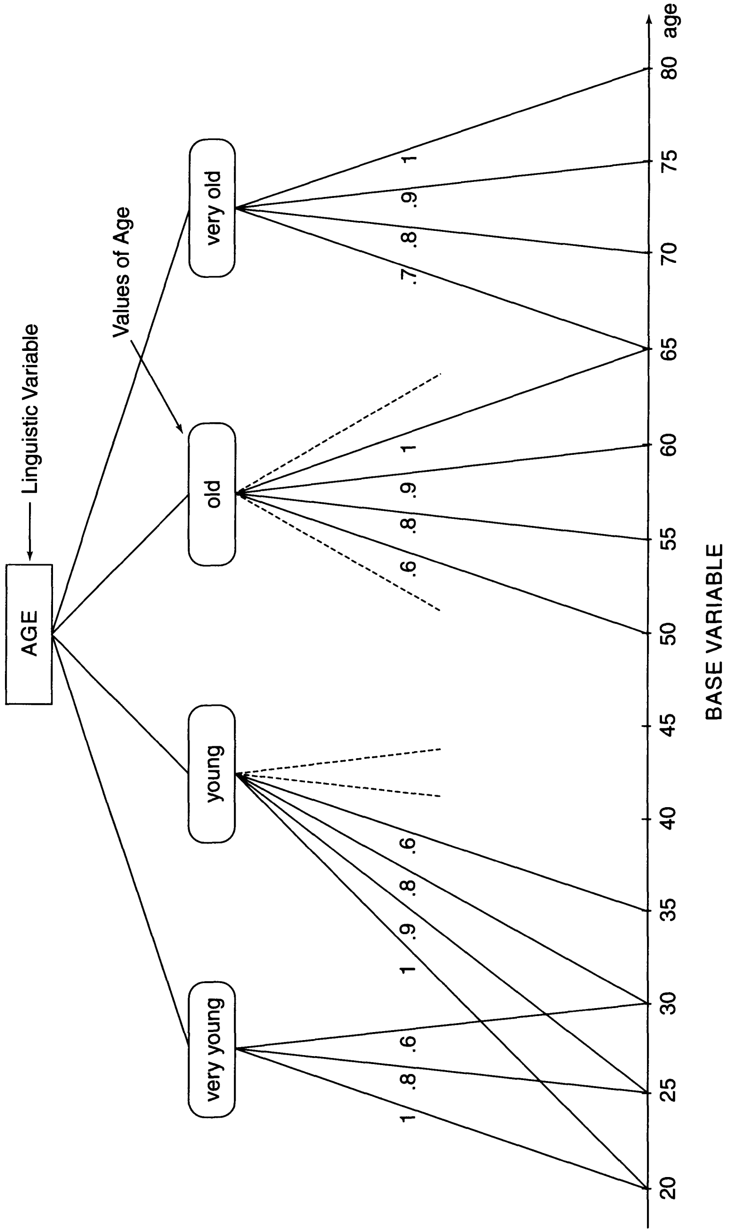


Figure 9–1. Linguistic variable "age."

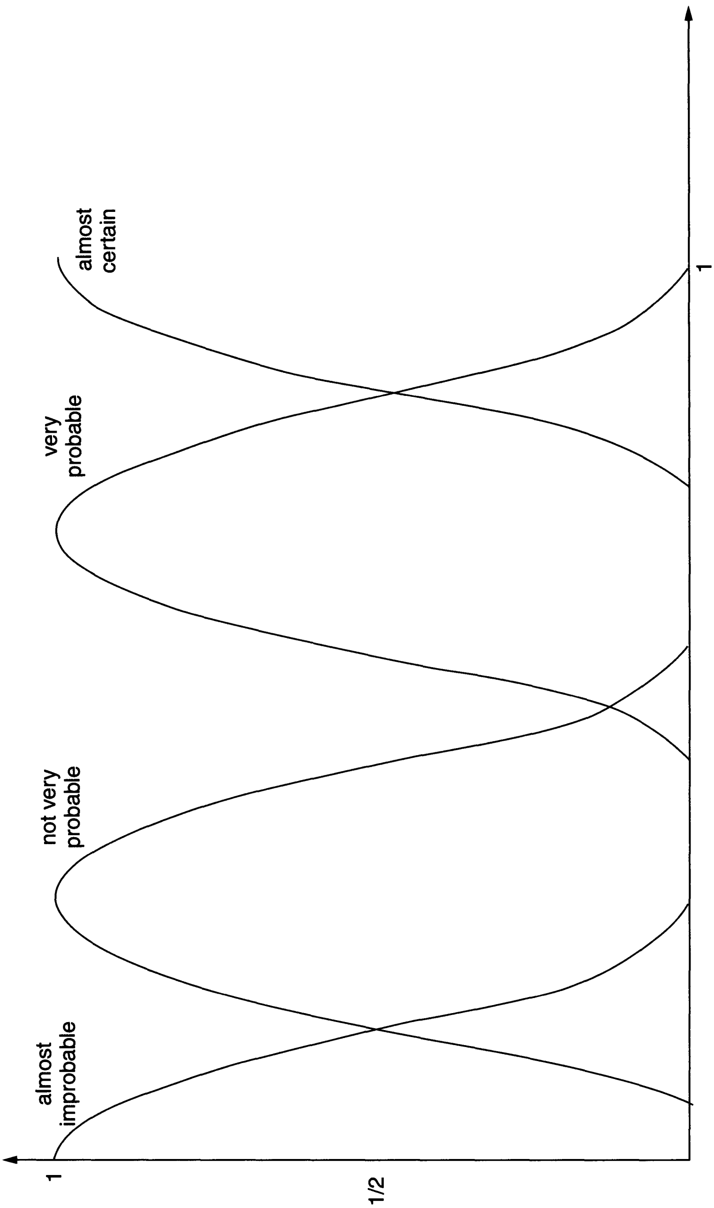


Figure 9–2. Linguistic variable “Probability.”

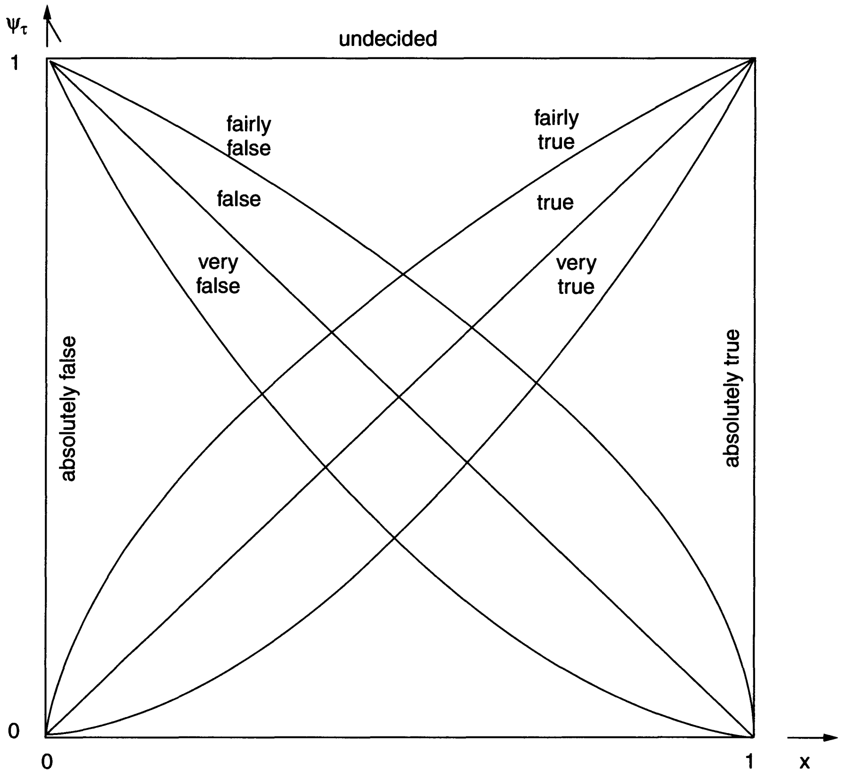


Figure 9–3. Linguistic variable "Truth."

The term set of the linguistic variable "Truth" has been defined differently by different authors. Baldwin [1979, p. 316] defines some of the terms as shown in figure 9–3. Here,

$$\mu_{\text{very true}}(\nu) = (\mu_{\text{true}}(\nu))^2 \qquad \nu \in [0,1]$$

$$\mu_{\text{fairly true}}(\nu) = (\mu_{\text{true}}(\nu))^{1/2} \qquad \nu \in [0,1]$$

and so on. Zadeh [1973a, p. 99] suggests for the term *true* the membership function

$$\mu_{\text{true}}(\nu) = \begin{cases} 0 & \text{for } 0 \le \nu \le a \\ 2 \cdot \left(\dfrac{\nu - a}{1 - a}\right)^2 & \text{for } a \le \nu \le \dfrac{a+1}{2} \\ 1 - 2 \cdot \left(\dfrac{\nu - 1}{1 - a}\right)^2 & \text{for } \dfrac{a+1}{2} \le \nu \le 1 \end{cases}$$

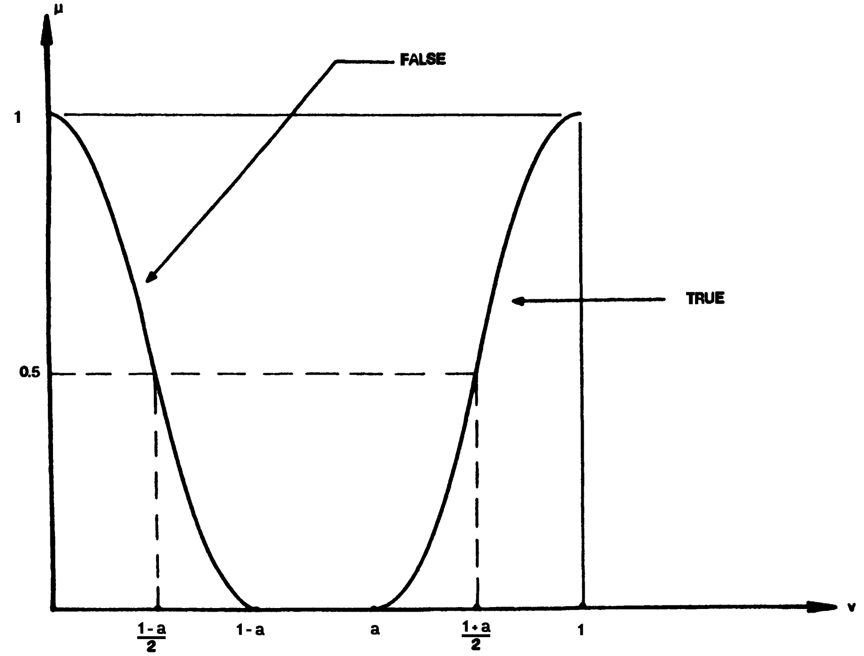


Figure 9–4. Terms "True" and "False."

where $\nu = (1 + a)/2$ is called the *crossover* point, and $a \in [0, 1]$ is a parameter that indicates the subjective judgment about the minimum value of ν in order to consider a statement as "true" at all.

The membership function of "false" is considered as the mirror image of "true," that is,

$$\mu_{\text{false}}(v) = \mu_{\text{true}}(1 - v) \quad 0 \leq v \leq 1$$

Figure 9–4 [Zadeh 1973a, p. 99] shows the terms *true* and *false*.

Of course, the membership functions of true and false, respectively, can also be chosen from the finite universe of truth values. The term set of the linguistic variable "Truth" is then defined as [Zadeh 1973a, p. 99]

$$T(\text{Truth}) = \{\text{true, not true, very true, not very true}, \ldots, \text{false, not false, very false}, \ldots, \text{not very true and not very false}, \ldots\}$$

The fuzzy sets (possibility distribution) of those terms can essentially be determined from the term *true* or the term *false* by applying appropriately the below-mentioned modifiers (hedges).

Definition 9–2

A linguistic variable x is called *structured* if the term set $T(x)$ and the meaning $\tilde{M}(x)$ can be characterized algorithmically. For a structured linguistic variable, $\tilde{M}(x)$ and $T(x)$ can be regarded as algorithms that generate the terms of the term set and associate meanings with them.

Before we illustrate this by an example, we need to define what we mean by a "hedge" or a "modifier."

Definition 9–3

A *linguistic hedge* or a *modifier* is an operation that modifies the meaning of a term or, more generally, of a fuzzy set. If $\tilde{A}$ is a fuzzy set, then the modifier m generates the (composite) term $\tilde{B} = m(\tilde{A})$.

Mathematical models frequently used for modifiers are as follows:

$$\text{concentration: } \mu_{\text{con}(\tilde{A})}(u) = (\mu_{\tilde{A}}(u))^2$$

$$\text{dilation: } \mu_{\text{dil}(\tilde{A})}(u) = (\mu_{\tilde{A}}(u))^{1/2}$$

contrast intensification:

$$\mu_{\text{int}(\tilde{A})}(u) = \begin{cases} 2(\mu_{\tilde{A}}(u))^2 & \text{for } \mu_{\tilde{A}}(u) \in [0, .5] \\ 1 - 2(1 - \mu_{\tilde{A}}(u))^2 & \text{otherwise} \end{cases}$$

Generally the following linguistic hedges (modifiers) are associated with above-mentioned mathematical operators:

If $\tilde{A}$ is a term (a fuzzy set), then

$$\begin{aligned} \text{very } \tilde{A} &= \text{con}(\tilde{A}) \\ \text{more or less } \tilde{A} &= \text{dil}(\tilde{A}) \\ \text{plus } \tilde{A} &= \tilde{A}^{1.25} \\ \text{slightly } \tilde{A} &= \text{int [plus } \tilde{A} \text{ and not (very } \tilde{A})] \end{aligned}$$

where "and" is interpreted possibilistically.

Example 9–2 [Zadeh 1973a, p. 83]

Let us reconsider from example 9–1 the linguistic variable "Age." The term set shall be assumed to be

$$T(\text{Age}) = \{\text{old, very old, very very old, . . . }\}$$

The term set can now be generated recursively by using the following rule (algorithm):

$$T^{i+1} = \{\text{old}\} \cup \{\text{very } T^i\}$$

that is,

$$\begin{aligned} T^0 &= \emptyset \\ T^1 &= \{\text{old}\} \\ T^2 &= \{\text{old, very old}\} \\ T^3 &= \{\text{old, very old, very very old}\} \end{aligned}$$

For the semantic rule, we only need to know the meaning of "old" and the meaning of the modifier "very" in order to determine the meaning of an arbitrary term of the term set. If one defines "very" as the concentration, then the terms of the term set of the structured linguistic variable "Age" can be determined, given that the membership function of the term "old" is known.

Definition 9–4 [Zadeh 1973a, p. 87]

A *Boolean linguistic variable* is a linguistic variable whose terms, X, are Boolean expressions in variables of the form X_p, $m(X_p)$ where X_p is a primary term and m is a modifier. $m(X_p)$ is a fuzzy set resulting from acting with m on X_p.

Example 9–3

Let "Age" be a Boolean linguistic variable with the term set

$$T(\text{Age}) = \{\text{young, not young, old, not old, very young, not young, and not old, young or old, } \ldots\}$$

Identifying " 'and" with the intersection, "or" with the union, "not" with the complementation, and "very" with the concentration, we can derive the meaning of different terms of the term set as follows:

$$\begin{aligned} \tilde{M}(\text{not young}) &= \neg\ \text{young} \\ \tilde{M}(\text{not very young}) &= \neg\ (\text{young})^2 \\ \tilde{M}(\text{young or old}) &= \text{young } \cup \text{ old etc.} \end{aligned}$$

Given the two fuzzy sets (primary terms)

$$\tilde{M}(\text{young}) = \{(u, \mu_{\text{young}}(u)) | u \in [0, 100]\}$$

where

$$\mu_{\text{young}}(u) = \begin{cases} 1 & u \in [0, 25] \\ \left(1 + \left(\dfrac{u-25}{5}\right)^2\right)^{-1} & u \in (25, 100] \end{cases}$$

and

$$\tilde{M}(\text{old}) = \{(u, \mu_{\text{old}}(u)) | u \in [0, 100]\}$$

where

$$\mu_{\text{old}}(u) = \begin{cases} 1 & u \in [0, 50] \\ \left(1 + \left(\dfrac{u-50}{5}\right)^{-2}\right)^{-1} & u \in (50, 100] \end{cases}$$

then the membership function of the term "young or old" would, for instance, be

$$\mu_{\text{young or old}}(u) = \begin{cases} 1 & \text{if} \quad u \in [0, 25] \\ \left(1 + \left(\dfrac{u-25}{5}\right)^2\right)^{-1} & \text{if} \quad u \in (25, 50] \\ \max\left\{\left(1 + \left(\dfrac{u-25}{5}\right)^2\right)^{-1}, \right. & \\ \left. \left(1 + \left(\dfrac{u-50}{5}\right)^{-2}\right)^{-1}\right\} & \text{if} \quad u \in (50, 100] \end{cases}$$

9.2 Fuzzy Logic

9.2.1 Classical Logics Revisited

Logics as bases for reasoning can be distinguished essentially by their three topic-neutral (context-independent) items: truth values, vocabulary (operators), and reasoning procedure (tautologies, syllogisms).

In Boolean logic, truth values can be 0 (false) or 1 (true), and by means of these truth values, the vocabulary (operators) is defined via truth tables.

Let us consider two statements, A and B, either of which can be true or false, that is, have the truth value 1 or 0. We can construct the following truth tables:

A	B	$\wedge$	$\vee$	$x\vee$	$\Rightarrow$	$\Leftrightarrow$	?
1	1	1	1	0	1	1	1
1	0	0	1	1	0	0	1
0	1	0	1	1	1	0	0
0	0	0	0	0	1	1	0

There are $2^{2^2} = 16$ truth tables, each defining an operator. Assigning meanings (words) to these operators is not difficult for the first 4 or 5 columns: the first obviously characterizes the "and," the second the "inclusive or," the third the "exclusive or," and the fourth and fifth the implication and the equivalence. We will have difficulties, however, interpreting the remaining nine columns in terms of our language. If we have three statements rather than two, this task of assigning meanings to truth tables becomes even more difficult.

So far it has been assumed that each statement, A and B, could clearly be classified as true or false. If this is no longer true, then additional truth values, such as "undecided" or a similar description, can and have to be introduced, which leads to the many existing systems of multivalued logic. It is not difficult to see how the above-mentioned problems of two-valued logic in "calling" truth tables or operators increase as we move to multivalued logic. For only two statements and three possible truth values, there are already $3^{3^2} = 729$ truth tables! The uniqueness of interpretation of truth tables, which is so convenient in Boolean logic, disappears immediately because many truth tables in three-valued logic look very much alike.

The third topic-neutral item of logical systems is the reasoning procedure itself, which is generally based on tautologies such as

modus ponens: $(A \wedge (A \Rightarrow B)) \Rightarrow B$
modus tollens: $((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$
syllogism: $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
contraposition: $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

Let us consider the modus ponens, which could be interpreted as: "If A is true and if the statement 'If A is true then B is true' is also true, then B is true."

The term *true* is used at different places and in two different senses: All but the last "trues" are material trues, that is, they are taken as a matter of fact, while the last 'true" is a topic-neutral logical "true." In Boolean logic, however, these "trues" are all treated the same way [see Mamdani and Gaines 1981, p. xv]. A distinction between material and logical (necessary) truth is made in so-called extended logics: Modal logic [Hughes and Cresswell 1968] distinguishes between necessary and possible truth, and tense logic between statements that were true in the past and those that will be true in the future. Epistemic logic deals with knowledge and belief and deontic logic with what ought to be done and what is permitted to be true. Modal logic, in particular, might be a very good basis for applying different measures and theories of uncertainty, as indicated in chapter 4.

Another extension of Boolean logic is predicate calculus, which is a set theoretic logic using quantifiers (all, etc.) and predicates in addition to the operators of Boolean logic.

Fuzzy logic [Zadeh 1973a, p. 101] is an extension of set-theoretic multivalued logic in which the truth values are linguistic variables (or terms of the linguistic variable truth).

Since operators, like $\vee$, $\wedge$, $\neg$, $\Rightarrow$ in fuzzy logic are also defined by using truth tables, the extension principle can be applied to derive definitions of the operators. So far, possibility theory (see section 8.1) has primarily been used in order to define operators in fuzzy logic, even though other operators have also been investigated (see, for instance, Mizumoto and Zimmermann [1982]), and could also be used. In this book, we will limit considerations to possibilistic interpretations of linguistic variables, and we will also stick to the original proposals of Zadeh [1973a]. To the interested reader, however, we suggest supplemental study of alternative approaches such as those by Baldwin [1979], Baldwin and Pilsworth [1980], Giles [1979, 1980], and others.

If $\nu(A)$ is a point in $V = [0, 1]$, representing the truth value of the proposition "u is A" or simply A, then the truth value of not A is given by

$$\nu(\text{not } A) = 1 - \nu(A)$$

Definition 9–5

If $\tilde{\nu}(A)$ is a normalized fuzzy set, $\tilde{\nu}(A) = \{(\nu_i, \mu_i) | i = 1, \ldots, n, \nu_i \in [0, 1]\}$, then by applying the extension principle, the *truth value of* $\tilde{\nu}$(*not* A) is defined as

$$\tilde{\nu}(\text{not } A) = \{(1 - \nu_i, \mu_i) | i = 1, \ldots, n, \nu_i \in [0, 1]\}$$

In particular, "false" is interpreted as "not true," that is,

$$\tilde{\nu}(\text{false}) = \{(1 - \nu_i, \mu_i) | i = 1, \ldots, n, \nu_i \in [0, 1]\}$$

Example 9–4

Let us consider the terms *true* and *false*, respectively, defined as the following possibility distributions:

$$\tilde{v}(\text{true}) = \{(.5, .6), (.6, .7), (.7, .8), (.8, .9), (.9, 1), (1, 1)\}$$
$$\tilde{v}(\text{false}) = \tilde{v}(\text{not true}) = \{(.5, .6), (.4, .7), (.3, .8), (.2, .9), (.1, 1), (0, 1)\}$$

Then

$$\tilde{v}(\text{very true}) = \{(.5, .36), (.6, .49), (.7, .64), (.8, .81), (.9, 1), (1, 1)\}$$
$$\tilde{v}(\text{very false}) = \{(.5, .36), (.4, .49), (.3, .64), (.2, .81), (.1, 1), (0, 1)\}$$

It has already been mentioned that fuzzy logic is essentially considered as an application of possibility theory to logic. Hence the logical operators "and," "or," and "not" are defined accordingly.

Definition 9–6

For numerical truth values $\nu(A)$ and $\nu(B)$, the logical operations *and*, *or*, *not*, and *implied* are defined as

$$\nu(A) \wedge \nu(B) = \nu(A \wedge B) = \min\{\nu(A), \nu(B)\}$$
$$\nu(A) \vee \nu(B) = \nu(A \vee B) = \max\{\nu(A), \nu(B)\}$$
$$\neg\nu(A) = 1 - \nu(A)\}$$
$$\nu(A) \Rightarrow \nu(B) = \nu(A \Rightarrow B) = \neg\nu(A) \vee \nu(B) = \max\{1 - \nu(A), \nu(B)\}$$

If

$$\tilde{v}(A) = \{(\nu_i, \alpha_i)\}, \quad \alpha_i \in [0,1], \nu_i \in [0,1]$$
$$\tilde{v}(B) = \{(w_j, \beta_j)\}, \quad \beta_i \in [0,1], \omega_j \in [0,1]$$
$$i = 1, \dots, n; j = 1, \dots, m$$

then

$$\tilde{v}(A \text{ and } B) = \tilde{v}(A) \wedge \tilde{v}(B) = \left\{\left(u = \min\{\nu_i, w_j\}, \quad \max_{u=\min\{\nu_i, \omega_j\}} \min\{\alpha_i, \beta_j\}\right)\right| \; i = 1, \dots, n; j = 1, \dots, m\right\}$$

(This is equivalent to the intersection of two type 2 fuzzy sets.) The other operators are defined accordingly.

Example 9–5

Let $\tilde{v}(A) = \text{true} = \{(.5, .6), (.6, .7), (.7, .8), (.8, .9), (.9, 1), (1, 1)\}$.
Then

$$\neg\tilde{v}(A) = \{(0, 1), (.1, 1), (.2, 1), (.3, 1), (.4, 1), (.5, .4), (.6, .3), (.7, .2), (.8, .1)\}$$

9.2.2 Linguistic Truth Tables

As mentioned at the beginning of this section, binary connectives (operators) in classical two- and many-valued logics are normally defined by the tabulation of truth values in truth tables. In fuzzy logic, the number of truth values is, in general, infinite. Hence tabulation of the truth values for operators is not possible. We can, however, tabulate truth values, that is, terms of the linguistic variable "Truth," for a finite number of terms, such as true, not true, very true, false, more or less true, and so on.

Zadeh [1973a, p. 109] suggests truth tables for the determination of truth values for operators using a four-valued logic including the truth values true, false, undecided, and unknown. "Unknown" is then interpreted as "true or false" ($T + F$), and "undecided" is denoted by $\ominus$.

Extending the normal Boolean logic with truth values true (1) and false (0) to a (fuzzy) three-valued logic (true = T, false = F, unknown = $T + F$), with a universe of truth values being two-valued (true and false), we obtain the following truth tables, in which the first column contains the truth values for a statement A and the first row those for a statement B [Zadeh 1973a, p. 116]:

$\wedge$	T	F	$T+F$
T	T	F	$T+F$
F	F	F	F
$T+F$	$T+F$	F	$T+F$

Truth table for "and"

$\vee$	T	F	$T+F$
T	T	F	T
F	T	F	$T+F$
$T+F$	T	$T+F$	$T+F$

Truth table for "or"

	$\neg$
T	F
F	T
$T+F$	$T+F$

Truth table for "not"

If the number of truth values (terms of the linguistic variable truth) increases, one can still "tabulate" the truth table for operators by using definition 9–6 as follows: Let us assume that the i^{th} row of the table represents "not true" and the j^{th} column "more or less true." The $(i, j)^{th}$ entry in the truth table for "and" would then contain the entry for "not true $\wedge$ more or less true." The resulting fuzzy set would, however, most likely not correspond to any fuzzy set assigned to the terms of the term set of "truth." In this case, one could try to find the fuzzy set of the term that is most similar to the fuzzy set resulting from the computations. Such a term would then be called *linguistic approximation*. This is an analogy to statistics, where empirical distribution functions are often approximated by well-known standard distribution functions.

Example 9–6

Let $V = \{0, .1, .2, \ldots, 1\}$ be the universe,
true = $\{(.8, .9), (.9, 1), (1, 1)\}$,
more or less true = $\{(.6, .2), (.7, .4), (.8, .7), (.9, 1), (1, 1)\}$, and
almost true = $\{(.8, .9), (.9, 1), (1, .8)\}$.

Let "more or less true" be the i^{th} row and "almost true" the j^{th} column of the truth table for "or."

Then "more or less true $\vee$ almost true" is the $(i, j)^{th}$ entry in the table:

$$\begin{aligned}&\text{more or less true} \vee \text{almost true}\\ &\quad = \{(.6, .2), (.7, .4), (.8, .7), (.9, 1), (1, 1)\} \vee \{(.8, .9), (.9, 1), (1, .8)\}\\ &\quad = \{(.6, .2), (.7, .4), (.8, .9), (.9, 1), (1, 1)\}\end{aligned}$$

Now we can approximate the right-hand side of this equation by

$$\text{true} = \{(.8, .9), (.9, 1), (1, 1)\}$$

This yields

$$\text{"more or less true} \vee \text{almost true"} \approx \text{"true."}$$

Baldwin [1979] suggests another version of fuzzy logic—fuzzy truth tables, and their determination: The truth values on which he bases his suggestions were shown graphically in figure 9–3. They were defined as

$$\begin{aligned}\text{true} &= \{(\nu, \mu_{\text{true}}(\nu) = \nu) | \nu \in [0, 1]\}\\ \text{false} &= \{(\nu, \mu_{\text{false}}(\nu) = 1 - \mu_{\text{true}}(\nu)) | \nu \in [0, 1]\}\\ \text{very true} &= \left\{\left(\nu, (\mu_{\text{true}}(\nu))^2\right) | \nu \in [0, 1]\right\}\\ \text{fairly true} &= \left\{\left(\nu, (\mu_{\text{true}}(\nu))^{1/2}\right) | \nu \in [0, 1]\right\}\\ \text{undecided} &= \{(\nu, 1) | \nu \in [0, 1]\}\end{aligned}$$

Very false and fairly false were defined correspondingly, and

$$\text{absolutely true} = \{(\nu, \mu_{at}(\nu)) | \nu \in [0, 1]\}$$

$$\text{with } \mu_{at}(\nu) = \begin{cases} 1 & \text{for } \nu = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{absolutely false} = \{(\nu, \mu_{af}(\nu)) | \nu \in [0, 1]\}$$

$$\text{with } \mu_{af}(\nu) = \begin{cases} 1 & \text{for } \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$(\text{very})^k \text{true} \rightarrow \text{absolutely true as } k \rightarrow \infty$$

$$(\text{very})^k \text{false} \rightarrow \text{absolutely false as } k \rightarrow \infty$$

$$(\text{fairly})^k \text{true} \rightarrow \text{undecided as } k \rightarrow \infty$$

$$(\text{fairly})^k \text{false} \rightarrow \text{undecided as } k \rightarrow \infty$$

Using figure 9–3 and the interpretations of "and" and "or" as minimum and maximum, respectively, the following truth table results [Baldwin 1979, p. 318]:

$\nu(P)$	$\nu(Q)$	$\nu(P \text{ and } Q)$	$\nu(P \text{ or } Q)$
false	false	false	false
true	false	false	true
true	true	true	true
undecided	false	false	undecided
undecided	true	undecided	true
undecided	undecided	undecided	undecided
true	very true	true	very true
true	fairly true	fairly true	true

Some more considerations and assumptions are needed to derive the truth table for the implication. Baldwin considers his fuzzy logic to rest on two pillars: the denumberably infinite multivalued logic system of Lukasiewicz logic and fuzzy set theory:

> Implication statements are treated by a composition of fuzzy truth value restrictions with a Lukasiewicz logic implication relation on a fuzzy truth space. Set theoretic considerations are used to obtain fuzzy truth value restrictions from conditional fuzzy linguistic statements using an inverse truth functional modification procedure. Finally true functions modification is used to obtain the final conclusion [Baldwin 1979, p. 309].

9.3 Approximate and Plausible Reasoning

We already mentioned that in traditional logic the main tools of reasoning are tautologies, such as, for instance, the modus ponens—that is $(A \wedge (A \Rightarrow B)) \Rightarrow B$ or

Premise	A is true
Implication	If A then B
Conclusion	B is true

Here A and B are crisply defined statements or propositions; the A's in the premise and the implication are identical, and so are the B's in the implication and conclusion. The "implication" is defined via truth tables, as shown in section 9.2.1.

Approximate and plausible reasoning are ways of drawing conclusions from hypotheses. They relax even more stringent assumptions of dual logic than fuzzy logic does and try to approach human reasoning even more closely.

Three natural generalizations of the classical modus ponens are

1. To modify the definition of the "implication,"
2. To allow statements that are no longer crisp but contain a fuzzy set, such as linguistic variables, and
3. To relax the identity of the A's and B's in the premise rule and conclusion by substituting for "identical" the term "similar."

Relaxations of point 2 lead to "approximate reasoning," and relaxations of points 2 and 3 lead to "plausible reasoning."

We shall first briefly consider point 1 and then turn to points 2 and 3.

The rule "if A then B" is often written as $A \rightarrow B$. The symbol "$\rightarrow$" is then often interpreted as implication, whose meaning is formally defined in logic. Obviously, there are two "translations" between the three different levels involved: the linguistic level (rule), the symbolic level ($\rightarrow$), and the formal logical level.

The relationship between the linguistic expression "if A then B" and the respective mathematical description cannot be derived formally, but only empirically. This problem belongs in the area of psycholinguistics, and empirical research in this direction is still very rare [Spiess 1989].

If "$A \rightarrow B$" is interpreted as material implication, in which A is called the premise and B the consequence, then the truth values $v(A)$, $v(B)$, and $v(A \rightarrow B)$ can in dual logic be either 0 or 1. As shown in the truth table in section 9.2.1, the truth value of $v(A \rightarrow B)$ is 0 if A is true and B is false; otherwise, its truth value is 1. This corresponds to the view that the implication is true whenever the consequence is at least as true as the premise. In Boolean logic, $A \rightarrow B$ is equivalent to $\neg A \vee (A \wedge B)$ (not A or (A and B)).

On the bases of these basic relationships, various implication operators have been defined. Ruan [1991] has investigated 18 of these definitions, which are all restricted to the min-max theory. We only show a selection of them in the next table. x denotes the degree of truth (or degree of membership) of the premise, y the respective values for the consequence, and I the resulting degree of truth for the implication.

Name	*Definition of Implication Operator*
Early Zadeh	$I_m(x, y) = \max(1 - x, \min(x, y))$
Lukasiewicz	$I_a(x, y) = \min(1, 1 - x + y)$
Minimum (Mamdani)	$I_c(x, y) = \min(x, y)$
Standard Star (Gödel)	$I_g(x, y) = \begin{cases} 1 & x \le y \\ y & \text{elsewhere} \end{cases}$
Kleene–Dienes	$I_b(x, y) = \max(1 - x, y)$
Gaines	$I_\Delta(x, y) = \begin{cases} 1 & x \le y \\ y/z & \text{elsewhere} \end{cases}$
Yager	$I_E(x, y) = y^x$

The "quality" of these implication operators could again be evaluated either empirically or axiomatically. For the latter, a well-accepted axiomatic system such as that of Smets and Magrez [1987] can be used. The authors assume that the implication operator is truth functional, i.e., that the truth of "A → B" only depends on the truth of A and B. They have formulated the following axioms:

1. $v(A \to B) = v(\neg B \to \neg A)$
   (contrapositive symmetry)
2. $v(A \to (B \to C)) = v(B \to (A \to C))$
   (exchange principle)
3. $v(A \to B) \ge v(C \to D)$ if
   $v(A) \le v(C)$ and/or $v(B) \ge v(D)$
   (monotonicity)
4. $v(A \to B) = 1$ if $v(A) \le v(B)$
   (boundary condition)
5. $v(T \to A) = v(A)$, where T stands for tautology
   (neutrality principle)
6. $v(A \to B)$ is continuous in its arguments
   (continuity)

Table 9–1 shows which of the implication operators satisfy (Y) or violate (N) the above axioms.

Table 9–1. Formal quality of implication operators.

	I_m	I_a	I_c	I_g	I_b	I	I_B
A1 Contraposition	N	Y	N	N	Y	N	N
A2 Exchange Principle	N	Y	Y	Y	Y	N	Y
A3 Monotonicity	N	Y	N	Y	Y	Y	Y
A4 Boundary Condition	N	Y	N	Y	N	Y	N
A5 Neutrality Principle	Y	Y	Y	Y	Y	Y	Y
A6 Continuity	Y	Y	Y	N	Y	N	N

If one uses the fraction of the axioms that are satisfied by the various implications as their degree of membership in the fuzzy set "good implication operators," then one would obtain the following fuzzy set:

Good Implication Operators

$$\left\{\left(I_m, \frac{1}{3}\right), (I_\alpha, 1), \left(I_c, \frac{1}{2}\right), \left(I_g, \frac{2}{3}\right), \left(I_b, \frac{5}{6}\right), \left(I_\Delta, \frac{1}{2}\right), \left(I_E, \frac{1}{2}\right)\right\}$$

For approximate and plausible reasoning as defined above, the modus ponens is extended to the "generalized modus ponens" [Zadeh 1973a, p. 56; Mizumoto et al. 1979; Mamdani 1977a].

Example 9–7

Let $\tilde{A}$, $\tilde{A}'$, $\tilde{B}$, $\tilde{B}'$ be fuzzy statements; then the generalized modus ponens reads

$$\begin{array}{rl} \text{Premise:} & x \text{ is } \tilde{A}' \\ \text{Implication:} & \text{If } x \text{ is } \tilde{A}, \text{ then } y \text{ is } \tilde{B} \\ \hline \text{Conclusion:} & y \text{ is } \tilde{B}' \end{array}$$

Premise: This tomato is very red.
Implication: If a tomato is red then the tomato is ripe.

Conclusion: This tomato is very ripe.

It should be mentioned, however, that the generalized modus ponens alone does not allow us to obtain conclusions from unequal premises. Such an inference presupposes or necessitates knowledge about modifications of the premises and their consequences (for example, knowledge that an increase in "redness" indicates an increase in "ripeness" [Dubois and Prade 1984b, p. 325].

In 1973, Zadeh suggested the compositional rule of inference for the abovementioned type of fuzzy conditional inference. In the meantime, other authors (for instance, Baldwin [1979]; Baldwin and Pilsworth [1980]; Baldwin and Guild [1980]; Mizumoto et al. [1979]; Mizumoto and Zimmermann [1982]; Tsukamoto [1979]), have suggested different methods and have also investigated the modus tollens, syllogism, and contraposition. In this book, however, we shall restrict considerations to Zadeh's compositional rule of inference.

Definition 9–7 [Zadeh 1973a, p. 148]

Let $\tilde{R}(x)$, $\tilde{R}(x, y)$, and $\tilde{R}(y)$, $x \in X$, $y \in Y$, be fuzzy relations in X, $X \times Y$, and Y, respectively, that act as fuzzy restrictions on x, (x, y), and y, respectively. Let $\tilde{A}$ and $\tilde{B}$ denote particular fuzzy sets in X and $X \times Y$. Then the *compositional rule of inference* asserts that the solution of the relational assignment equations (see definition 8–1) $\tilde{R}(x) = \tilde{A}$ and $\tilde{R}(x, y) = \tilde{B}$ is given by $\tilde{R}(y) = \tilde{A} \circ \tilde{B}$, where $\tilde{A} \circ \tilde{B}$ is the composition of $\tilde{A}$ and $\tilde{B}$.

Example 9–8

Let the universe be $X = \{1, 2, 3, 4\}$.
$\tilde{A}$ = little = $\{(1, 1), (2, .6), (3, .2), (4, 0)\}$.
$\tilde{R}$ = "approximately equal" be a fuzzy relation defined by

$\tilde{R}$:	1	2	3	4
1	1	.5	0	0
2	.5	1	5	0
3	0	.5	1	.5
4	0	0	.5	1

For the formal inference, denote

$$\tilde{R}(x) = \tilde{A}, \quad \tilde{R}(x, y) = \tilde{B}, \quad \text{and} \quad \tilde{R}(y) = \tilde{A} \circ \tilde{B}$$

Applying the max-min composition for computing $\tilde{R}(y) = \tilde{A} \circ \tilde{B}$ yields

$$\begin{aligned} \tilde{R}(y) &= \max \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R}}(x, y)\} \\ &= \{(1, 1), (2, .6), (3, .5), (4, .2)\} \end{aligned}$$

A possible interpretation of the inference may be the following:

x is little
x and y are approximately equal

---

y is more or less little

A direct application of approximate reasoning is the fuzzy algorithm (an ordered sequence of instructions in which some of the instructions may contain labels of fuzzy sets) and the fuzzy flow chart. We shall consider both in more detail in chapter 10. Here, however, we shall briefly describe fuzzy (formal) languages.

9.4 Fuzzy Languages

Fuzzy languages are formal languages based on fuzzy logic and approximate reasoning. Several of them have been developed by now, such as LPL [Adamo 1980], FLIP [Giles 1980], Fuzzy Planner [Kling 1973], and others. They are based on LP1, FORTRAN, LISP, and other programming languages and differ in their content as well as their aims. Here we shall sketch a meaning-representation language developed by Zadeh [Zadeh 1981a].

PRUF (acronym for *P*ossibilistic *R*elational *U*niversal *F*uzzy) is a meaning-representation language for natural languages and is based on possibility theory. PRUF may be employed as a language for the presentation of imprecise knowledge and as a means of making precise the fuzzy propositions expressed in a natural language. In essence, PRUF bears the same relationship to fuzzy logic that predicate calculus does to two-valued logic. Thus it serves to translate a set of premises expressed in natural language into expressions in PRUF to which the rules of inference of fuzzy logic or approximate reasoning may be applied. This yields other expressions in PRUF that can then be retranslated into natural language and become the conclusions inferred from the original premises.

The main constituents of PRUF are

1. a collection of translation rules, and
2. a set of rules of inference.

The latter corresponds essentially to fuzzy logic and approximate reasoning, as described in sections 9.2 and 9.3. The former will be described in more detail after the kind of representation in PRUF has been described and some more definitions introduced.

In definition 8–2, the relational assignment equation was defined. In PRUF, a possibility distribution π_x is assigned via the

$$\text{possiblility assignment equation (PAE): } \pi_x \triangleq \tilde{F}$$

to the fuzzy set $\tilde{F}$. The PAE corresponds to a proposition of the form "N is $\tilde{F}$" where N is the name of a variable, a fuzzy set, a proposition, or an object. For simplicity, the PAE will be written as in chapter 8 as

$$\pi_x = \tilde{F}$$

Example 9–9

Let N be the proposition "Peter is old"; then N (the variable) is called "Peter," $X \in [0, 100]$ is the linguistic variable "Age," "old" is, for instance, a term of the term set of "Age," and

$$\text{Peter is old} \rightarrow \pi_{\text{Age(Peter)}} = \text{old}$$

where $\rightarrow$ stands for "translates into."
There are two special types of possibility distributions that will be needed later.

Definition 9–8

The possibility distributions π_I with

$$\pi_I(u) = 1 \quad \text{for} \quad u \in U$$

is called the *unity possibility distribution* π_I, and with

$$\pi_\perp(\nu) = \nu \quad \text{for} \quad \nu \in [0, 1]$$

is defined the *unitary possibility distribution function* [Zadeh 1981a, p. 10].

In chapter 6 (definition 6–4), the projection of a binary fuzzy relation was defined. This definition holds not only for binary relations and numerical values of the related variables but also for linguistic variables.

Different fuzzy relations in a product space $U_1 \times U_2 \times \ldots \times U_n$ can have identical projections on $U_{i_1} \times \ldots \times U_{i_k}$. Given a fuzzy relation $\tilde{R}_q$ in $U_{i_1} \times \ldots \times U_{i_k}$, there exists, however, a unique relation $\tilde{R}_{qL}$ that contains all other relations whose projection on $U_{i_1} \times \ldots \times U_{i_k}$ is $\tilde{R}_q$. $\tilde{R}_{qL}$ is then called the cylindrical extension of $\tilde{R}_q$; the latter is the basis of $\tilde{R}_{qL}$ (see definitions 6–4, 6–5).

In PRUF, the operation "particularization" is also important: "By the particularization of a fuzzy relation or a possibility distribution which is associated with a variable $\tilde{X} \triangleq (\tilde{X}_1, \ldots, \tilde{X}_n)$, is meant the effect of specification of the possibility distributions of one or more subvariables (terms) of $\tilde{X}$. Particularization in PRUF may be viewed as the result of forming the conjunction of a proposition of the form "$\tilde{X}$ is $\tilde{F}$," where $\tilde{X}$ is an n-ary variable with particularizing propositions of the form "$\tilde{X}_s = \tilde{G}$," where $\tilde{X}_s$ is a subvariable (term) of $\tilde{X}$ and $\tilde{F}$ and $\tilde{G}$, respectively, are fuzzy sets in $U_1 \times U_2 \times \ldots U_n$ and $U_{i_1} \times \ldots \times U_{i_k}$, respectively" [Zadeh 1981a, p. 13].

Definition 9–9 [Zadeh 1981a, p. 13]

Let $\tilde{\pi}_X \triangleq \tilde{\pi}(X_1 \ldots X_n) = \tilde{F}$ and $\tilde{\pi}_X \triangleq \tilde{\pi}(X_{i_j}, \ldots, X_{i_k}) = \tilde{G}$ be possibility distributions induced by the propositions "$\tilde{X}$ is $\tilde{F}$" and "$\tilde{X}_s$ is $\tilde{G}$," respectively. The *particularization* of $\tilde{\pi}_X$ by $\tilde{X}_s = \tilde{G}$ is denoted by $\tilde{\pi}_X(\tilde{\pi}_{X_s} = \tilde{G})$ and is defined as the intersection of $\tilde{F}$ and $\tilde{G}$, that is,

$$\tilde{\pi}_x(\tilde{\pi}_s = \tilde{G}) = \tilde{F} \cap \tilde{G}'$$

where $\tilde{G}'$ is the cylindrical extension of $\tilde{G}$.

Example 9–10

Consider the proposition "Porsche is an attractive car," where attractiveness of a car as a function of mileage and top speed is defined in the following table.

Attractive cars	*Top speed (mph)*	*Mileage (mpg)*	μ
	60	30	.4
	60	35	.5
	60	40	.6
	70	30	.7
	85	25	.7
	90	25	.8
	95	25	.9
	100	20	1.0
	110	15	1.0

A particularizing proposition is "Porsche is a fast car," in which "fast" is defined in the following table:

Fast cars	*Top speed (mph)*	μ
	60	.4
	70	.6
	85	.7
	90	.8
	95	.9
	100	.95
	110	1.0

"Porsche is an attractive car" can equivalently be written as "Porsche is a fast car," that is, "Top speed (Porsche) is high" and "mileage (Porsche) is high."

Using definition 9–9, the particularized relation *attractive* (π_{speed} = Fast) can readily be computed, as shown in the next table:

Attractive cars	*Top speed*	*Mileage*	μ
	60	30	.4
	60	35	.4
	60	40	.4
	70	30	.6
	85	25	.7
	90	25	.8
	95	25	.9
	100	20	0.95
	110	15	1

Translation Rules in PRUF. The following types of fuzzy expressions will be considered:

1. Fuzzy propositions such as "All students are young," "*X* is much larger than *Y*," and "If Hans is healthy then Biggi is happy."
2. Fuzzy descriptors such as tall men, rich people, small integers, most, several, or few.
3. Fuzzy questions.

Fuzzy questions are reformulated in such a way that additional translation rules for questions are unnecessary. Questions such as "How *A* is *B*?" will be expressed in the form "*B* is ?*A*," where *B* is the body of the question and "?*A*" indicates the form of an admissible answer, which can be a possibility distribution (indicated as π); a truth value (indicated as τ); a probability value (indicated as λ); or a possibility value (indicated as ω).

The question "How tall is Paul?" to which a possibility distribution is expected as an answer, is phrased "Paul is ?π" (rather than "How tall is Paul ?π). "Is it true that Katrin is pretty?" would then be expressed as "Katrin is pretty ?τ" and "Where is the car ?w" as "The car is ?w."

PRUF is an intentional language, that is, an expression in PRUF is supposed to convey the intended rather than the literal meaning of the corresponding expression in a natural language. Transformations of expressions are also intended to be meaning-preserving. Translation rules are applied singly or in combination to yield an expression, *E*, in PRUF that is a translation of a given expression, *e*, in a natural language.

The most important basic categories of translation rules in PRUF are

Type I	Rules pertaining to modification
Type II	Rules pertaining to composition
Type III	Rules pertaining to quantification
Type IV	Rules pertaining to qualification

Examples of propositions to which these rules apply are the following [Zadeh 1981a, p. 29]:

Type I	*X* is very small.
	X is much larger than *Y*.
	Eleanor was very upset.
	The man with the blond hair is very tall.
Type II	*X* is small and *Y* is large. (conjunctive composition)
	X is small or *Y* is large. (disjunctive composition)
	If *X* is small, then *Y* is large. (conditional composition)
	If *X* is small, then *Y* is large else *Y* is very large. (conditional and conjunctive composition)
Type III	Most Swedes are tall.
	Many men are much taller than most men.
	Most tall men are very intelligent.
Type IV	Abe is young is not very true. (truth qualification)
	Abe is young is quite probable. (probability qualification)
	Abe is young is almost impossible. (possibility qualification)

Rules of Type I

Type I rules concern the modification of fuzzy sets representing propositions by means of hedges or modifiers (see definition 9–3).

If the proposition

$$P \triangleq N \text{ is } \tilde{F}$$

translates into the possibility assignment equation

$$\pi_{(X_1,\ldots,X_n)} = \tilde{F}$$

then the translation of the modified proposition

$$P^+ \triangleq N \text{ is } m\tilde{F} \text{ is}$$

$$\pi_{(X_1,\ldots,X_n)} = \tilde{F}^+$$

where $\tilde{F}^+$ is a modification of $\tilde{F}$ by the modifier m. As mentioned in chapter 9.1, the modifier "very" is defined to be the squaring operation, "more or less" the dilation, and so on.

Example 9–11

Let p be the proposition "Hans is old," where "old" may be the fuzzy set defined in example 9–1. The translation of $p^+ \triangleq$ "Hans is very old," assuming "very" to be modeled by squaring, would then be

$$\pi_{\text{Age(Hans)}} = (\text{old})^2 = \{(u, \mu_{(\text{old})^2}(u)) | u \in [0, 100]\}$$

where

$$\mu_{(\text{old})^2}(u) = \begin{cases} 0 & u \in [0, 50] \\ \left(\left(1 + \left(\dfrac{u-50}{5}\right)^{-2}\right)^{-1}\right)^2 & u \in (50, 100] \end{cases}$$

Rules of Type II

Rules of type II translate compound statements of the type

$$p = q * r$$

where $*$ denotes a logical connective—for example, and (conjunction) or (disjunction), if . . . then (implication), and so on. Here, essentially the definitions of connectives defined in section 9.1 and 9.2 are used in PRUF.

If the statements q and r are

$$q \hat{=} M \quad \text{is} \quad \tilde{F} \to \pi_{(X_1,\ldots,X_n)} = \tilde{F}$$
$$r \hat{=} N \quad \text{is} \quad \tilde{G} \to \pi_{(Y_1,\ldots,Y_n)} = \tilde{G}$$

then

$$(M \quad \text{is} \quad \tilde{F}) \text{ and } (N \quad \text{is} \quad \tilde{G}) \to \pi_{(X_1,\ldots,X_n,Y_1,\ldots,Y_n)} = \tilde{F} \times \tilde{G}$$

where

$$\tilde{F} \times \tilde{G} = \{((u,v), \mu_{\tilde{F}\times\tilde{G}}(u,v)) | u \in U, v \in V\}$$

and

$$\mu_{\tilde{F}\times\tilde{G}}(u,v) = \min\{\mu_F(u), \mu_G(v)\}$$

"If M is $\tilde{F}$, then N is $\tilde{G}$" $\to \pi_{(X_1,\ldots,X_n,Y_1,\ldots,Y_n)} = \tilde{F}'_L \oplus \tilde{G}'_L$ where $\tilde{F}'_L$ and $\tilde{G}'_L$ are the cylindrical extensions of $\tilde{F}$ and $\tilde{G}$ and $\oplus$ is the bounded sum defined in definition 3–9. Hence

$$\mu_{\tilde{F}'_L\oplus\tilde{G}'_L}(u,v) = \min\{1, \mu_{\tilde{F}}(u) + \mu_{\tilde{G}}(v)\}$$

Example 9–12 [Zadeh 1981a, pp. 32–33]

Assume that $u = v = 1, 2, 3$ and $M \hat{=} X$, $N \hat{=} Y$, and

$$\tilde{F} \hat{=} \text{small} \hat{=} \{(1,1),(2,.6),(3,.1)\}$$
$$\tilde{G} \hat{=} \text{large} \hat{=} \{(1,.1),(2,.6),(3,1)\}$$

Then X is small and Y is large $\to$

$$\pi(x,y) = \{[(1,1),.1],[(1,2),.6],[(1,3),1],[(2,1),.1],[(2,2),.6],$$
$$[(2,3),.6],[(3,1),.1],[(3,2),.1],[(3,3),.1]\}$$

X is small or Y is large $\to$

$$\pi(x,y) = \{[(1,1),1],[(1,2),1],[(1,3),1],[(2,1),.6],[(2,2),.6],$$
$$[(2,3),1],[(3,1),.1],[(3,2),.6],[(3,3),.1]\}$$

If X is small, then Y is large $\to$

$$\pi(x,y) = \{[(1,1),1],[(1,2),.6],[(1,3),1],[(2,1),.5],[(2,2),1],$$
$$[(2,3),1],[(3,1),1],[(3,2),1],[(3,3),1]\}$$

Translation rules of type II can, of course, also be applied to propositions containing linguistic variables. In some applications, it is convenient to represent fuzzy relations as tables (such as those shown in section 6.1). These tables can also be processed in PRUF.

Rules of Type III

Type III translation rules pertain to the translation of propositions of the form

$$P \triangleq QN \quad \text{are} \quad \tilde{F}$$

where N may also be a fuzzy set and Q is a so-called quantifier, for example, a term such as most, many, few, some, and so on. Examples are

Most children are cheerful.
Few lazy boys are successful.
Some men are much richer than most men.

A quantifier, Q, is in general a fuzzy set of which the universe is either the set of integers, the unit interval, or the real line.

Some quantifiers, such as most, many, and so on, refer to propositions of sets that may either be crisp or fuzzy. In this case, the definition of a quantifier makes use of the cardinality or the relative cardinality, as defined in definition 2–5.

In PRUF, the notation prop $(\tilde{F}/\tilde{G})$ is used to express the proportion of $\tilde{F}$ in $\tilde{G}$ where

$$\text{prop}(\tilde{F}/\tilde{G}) = \frac{\text{count}(\tilde{F} \cap \tilde{G})}{\text{count } \tilde{G}} = \frac{|\tilde{F} \cap \tilde{G}|}{|\tilde{G}|}$$

where "count" corresponds to the above-mentioned cardinality. The quantifier "most" may then be a fuzzy set

$$\tilde{Q} = \{[\text{prop}\,(\tilde{F}/\tilde{G}), \mu_{\text{most}}(u, v)] | u \in \tilde{F}, v \in \tilde{G}\}$$

Example 9–13

The quantifier "several" could, for instance, be represented by

$$\tilde{Q} \triangleq \text{several} \triangleq \{(3, .3), (4, .6), (5, 1), (6, .8), (7, .6), (8, .3)\}$$

Rules of Type IV

In PRUF, the concept of truth serves to make statements about the relative truth of a proposition p with respect to another reference proposition (and not with respect to reality!). Truth is taken to be a linguistic variable, as defined in section 9.1. Truth is then interpreted as the consistency of proposition p with proposition q. If

$$p \triangleq N \quad \text{is} \quad F \rightarrow \pi_p = \tilde{F}$$
$$q \triangleq N \quad \text{is} \quad G \rightarrow \pi_q = \tilde{G}$$

then the consistency of p with q is given as

$$\begin{aligned}\text{cons}\{N \quad \text{is} \quad F|N \quad \text{is} \quad G\} &\hat{=} \text{poss}\{N \quad \text{is} \quad F|N \quad \text{is} \quad G\} \\ &= \sup_{u \in U}\{\min(\mu_{\tilde{F}}(u), \mu_{\tilde{G}}(u))\}\end{aligned}$$

Example 9–14

Let

$$\begin{aligned} p &\hat{=} N \text{ is a small integer} \\ q &\hat{=} N \text{ is not a small integer}\end{aligned}$$

where

$$\text{small integer} \hat{=} \{(0,1),(1,1),(2,.8),(3,.6),(4,.5),(5,.4),(6,.2)\}$$

Then

$$\begin{aligned}\text{cons}\{p \mid q\} &= \sup\{[0,0,.2,.4,.5,.4,.2]\} \\ &= .5\end{aligned}$$

More in line with fuzzy set theory is the consideration of the truth of a proposition as a fuzzy number. Therefore Zadeh defines in the context of PRUF truth as follows:

Definition 9–10 [Zadeh 1981a, p. 42]

Let p be a proposition of the form "N is $\tilde{F}$," and let r be a reference proposition, $r \hat{=} N$ is $\tilde{G}$, where $\tilde{F}$ and $\tilde{G}$ are subsets of U. Then the truth, τ, of p relative to r is defined as the *compatibility of r with p*, that is,

$$\begin{aligned}\tau \hat{=} Tr(N \quad \text{is} \quad \tilde{F}|N \quad \text{is} \quad \tilde{G}) &\hat{=} \text{comp}(N \quad \text{is} \quad \tilde{G} \mid N \quad \text{is} \quad \tilde{F}) \\ &\hat{=} \mu_{\tilde{F}}(\tilde{G}) \\ &\hat{=} \{(\tau, \mu_{\tilde{F}}(\tilde{G}))|\tau \in [0,1]\}\end{aligned}$$

with

$$\mu_{\tilde{F}}(\tilde{G}) = \inf_{\tau \in [0,1]}\{\mu_{\tilde{F}}(u), \mu_{\tilde{G}}(u)\}, u \in U$$

The rule for truth qualification in PRUF can now be stated as follows [Zadeh 1981a, p. 44]: Let p be a proposition of the form

$$p \hat{=} N \quad \text{is} \quad \tilde{F}$$

and let q be a truth-qualified version of p expressed as

$$q \triangleq N \quad \text{is} \quad \tilde{F} \quad \text{is} \quad \tau$$

where τ is a linguistic truth value. q is semantically equivalent to the reference proposition, that is,

$$N \quad \text{is} \quad \tilde{F} \text{ is} \quad \tau \rightarrow N \quad \text{is} \quad \tilde{G}$$

where $\tilde{F}$, $\tilde{G}$, and τ are related by

$$\tau = \mu_{\tilde{F}}(\tilde{G})$$

In analogy to truth qualification, translation rules for probability qualification and possibility qualification have been developed in PRUF.

Example 9–15

Let

$$\begin{aligned} U &= N_0 = \{0, 1, 2, \ldots\}, \quad N \in N_0 \\ p &= N \quad \text{is} \quad \text{small} \\ r &= N \quad \text{is} \quad \text{approximately } 4 \end{aligned}$$

where

small = {(0, 1), (1, 1), (2, .8), (3, .6), (4, .4), (5, .2)}
approximately 4 = {(1, .1), (2, .2), (3, .5), (4, 1), (5, .5), (6, .2), (7, .1)}

Then

$$\begin{aligned} \tau &= Tr(N \quad \text{is} \quad \text{small}|N \quad \text{is} \quad \text{approximately } 4) \\ &= \text{comp}(N \quad \text{is} \quad \text{approximately } 4|N \quad \text{is} \quad \text{small}) \\ &= \{(\mu_{\text{small}}(u), \mu_4(u))|u \in U\} \\ &= \{(0, .2), (.2, .5), (.4, 1), (.6, .5), (.8, .2), (1, .1)\} \end{aligned}$$

9.5 Support Logic Programming and Fril

9.5.1 Introduction

Fril is a logic programming style implementation of support logic programming [Baldwin 1986, 1987, 1993]. It is a complete programming system with an incremental compiler, on-line help, a step-by-step debugger, modular code development, and optimization [Baldwin, Martin, and Pilsworth 1995]. It is written in C and is a Prolog system if no uncertainties are used. The style of programming can include the object-oriented paradigm by introducing the concept of a fuzzy object. A menu-driven window environment with dialogue boxes can be written in Fril to provide the intelligent systems application with a friendly front end. Fril can also be linked to Mathematica [Wolfram 1993], allowing mathematical equations to be solved as part of the inference process. Mathematical commands can be sent from Fril directly to Mathematica, and answers received by Fril can act as data for part of some inference process.

Fril is an ideal language for soft computing, since it is an efficient general logic programming language with special structures to handle uncertainty and imprecision. Four types of rules are allowed in Fril:

1. Prolog style rule
2. Probabilistic fuzzy rule
3. Causal relational rule
4. Evidential logic rule

The popularity and success of fuzzy control, which uses simple IF . . . THEN rules, should motivate knowledge engineers to investigate the use of Fril and fuzzy methods for intelligent systems. We would expect areas of application such as expert systems for large-scale engineering systems, vision-understanding systems, planning, robotics, military systems, medical and engineering diagnosis, economic planning, human interface systems, and data compression to benefit from this more general modeling approach.

The fuzzy sets representing possible feature values and the importance given to these features can be automatically derived from a data set of examples. The rules derived in this way provide a generalization of the specific instances given in the data set. This, along with the Fril inference rules, provides a theory of generalization and decision suitable for machine intelligence.

9.5.2 Fril Rules

The three Fril rules are of the form:

<head> IF *<body>* : *<list of support pairs>*

where the head of the rule can contain a fuzzy set. In the case of rules of types II and III, the body of the rule can be a conjunction of terms, a disjunction of terms, or a mixture of the two, and each term can contain a fuzzy set. The body of the fourth rule is a list of weighted features, where a feature is simply a condition that may contain a fuzzy set or the head of another rule. The list of support pairs provides intervals containing conditional probabilities of some instantiation of the head given some instantiation of the body.

An example of each type of rule is as follows:

Example 9–16: Rule of Type II

((suitability place X for sports stadium Y is *high*)
(access X from other parts of city is *easy*) (cost_to_build Y at X is *fairly cheap*)): [0.9, 1]

This rule states that there is a high probability that any place X is highly suitable to build a sports stadium Y if X is easily accessed and Y can be built fairly cheaply at X.

Example 9–17: Rules of Type III

((shoe_size man X is large)
((height X is tall) (height X is average) (height X is small)): [0.8, 1]
[0.5, 0.6] [0, 0.1]

This rule states that the probability of a tall man wearing large shoes is greater than 0.8. The probability that a man of average height wears large shoes is between 0.5 and 0.6. The probability that a small man wears large shoes is less than 0.1.

We can think of the rule as representing the relationship between two variables, S and H, where S is shoe size and H is height of man. S is instantiated to *large*, while H has three instantiations in the body of the rule. The rule expresses $Pr(S \text{ is } large|H \text{ is } h_i)$ where h_i is a particular fuzzy instantiation of H. This type of rule is useful to represent fuzzy causal nets and many other types of applications.

Example 9–18: Rules of Type IV

((suitability_as_secretary person X is good)
(evlog *most*((readability handwriting of X, *high*) 0.1
(neatness(X, *fairly good*)) 0.1
(qualifications X, *applicable*) 0.2
(concentration X, *long*) 0.1
(typing_skills X, *very good*) 0.3
(shorthand X, *adequate*) 0.2))): [1, 1] [0, 0]

This rule says that a person's suitability as a secretary is good if most of the weighted features in the body of the rule are satisfied. The term "most" is a fuzzy set that is chosen to provide optimism for those persons who satisfy the criteria well and pessimism for those who satisfy the criteria badly. Type III rules are evidential logic rules and can be used for vision understanding, classification, and case-based reasoning. The satisfaction of features such as (qualifications X

applicable) is determined from another rule with satisfaction as head. Methods can be used to determine near optimal weights and the fuzzy sets in the body of the rules from a data set of examples [Baldwin 1994]. These are discussed below.

Meta Rules

Types III and IV rules can be written in terms of types I and II rules. Other rules, which we can call meta rules, can be similarly defined in Fril.

9.5.3 Inference Methods in Fril

Consider a statement such as

most tall persons wear *large* shoes

The words printed in italics are fuzzy sets representing the vagueness of the definitions of these concepts.

This sentence can be replaced by the equivalent statement

$$Pr(\text{a person } X \text{ wears large shoes} | X \text{ is tall}) \geq 0.95$$

if we interpret "most" as the fuzzy set "greater_than_95%." We can simplify further if we replace the fuzzy set "greater_than_95%" with the support pair [0.95, 1], where a support pair is an interval containing a probability.

This could be written as a Fril rule:

((shoe_size of X large)
(height of X tall)): [0.95, 1]

The discrete fuzzy set *large* defined on the size domain and the continuous fuzzy set *tall* defined on the height are represented as list structures in Fril. For example,

```
set (height_domain (4 8))
set (size_domain (4 5 6 7 8 9 10 11 12 13))
(tall [5.8: 0, 6: 1] height_domain)
(large {9: 0.3, 10: 0.5, 11: 0.9, 12: 1, 13: 1} size_domain)
```

The height domain is all heights in the range [4 ft, 8 ft], and the size domain is the list of shoe sizes {4 5 6 7 8 9 10 11 12 13}. The membership of elements in the discrete fuzzy set are given to the right of the colon. For the continuous fuzzy set, the membership is 0 for all heights in the height domain smaller than 5.8 and 1 for all heights in the height domain larger than 6, and linear interpolation is used to determine the membership value for heights in the range [5.8, 6].

Assume we know the facts

((height of John average))

where the fuzzy set average is defined using the Fril statement

(average [5.8: 0, 5.9: 1, 6: 0] height_domain)

Then we should be able to conclude something like shoe_size of John is more_or_less_fairly_large. We would like to be able to provide an estimate from the fuzzy set conclusion for the size of John's shoes. This corresponds to defuzzifying the fuzzy set conclusion. We would only defuzzify if asked for a precise value.

> How can we determine the fuzzy set f for the conclusion
> ((shoe_size of X f))
> and how can we defuzzify this conclusion to give us the conclusion
> ((shoe_size of John s))
> corresponding to defuzzified value s?

The term in the body of the rule (height of X tall) is matched to (height of John tall) with X instantiated to John. There is only a partial match because average only partially matches the term "tall." The mass assignment theory allows us to determine an interval containing the conditional probability

$$Pr\{\text{(height of John tall)}|\text{(height of John average)}\}$$

This interval can be denoted by $[x_1, x_2]$. The process of determining this interval is called interval semantic unification. Fril automatically determines this interval. There is also a point-version semantic unification in which a point value is determined by intelligent filling in for unknown information. A query can be asked in Fril such that point semantic unification is used. In this case, Fril returns

$$Pr\{\text{(height of John tall)}|\text{(height of John average)}\} = x$$

We now know that the body of the rule is satisfied with a belief or probability given by the support pair $[x_1, x_2]$ or point value x. x_1 gives the necessary support for the body of the rule, and x_2 gives the possible support for the body of the rule. $1 - x_2$ gives the necessary support against the body of the rule being satisfied. We can now use an interval version of Jeffrey's rule of inference to determine a support pair for the consequence of the rule [Baldwin 1991]. Jeffrey's rule is of the form

$$Pr'(h) = \sum_{i=1}^{N} Pr(h|b_i)Pr'(b_i)$$

where $\{Pr(h|b_i\}$ represent conditional probabilities determined from a population of objects and $\{P'r(b_i)\}$ are probabilities or beliefs about a given object from the population. These primed probabilities are not determined with reference to the population of objects. The primed probabilities are specific to the one object under investigation. To make this more clear, consider the following example. From past observations and examination results, it is known that in a given school 90% of hardworking students obtain good passes in their final examinations. The probability Pr(good pass|hardworking) is obtained from population considerations. Consider a new boy to the school. By interviewing the body and from references, we estimate a belief that this boy will be hardworking, say, 0.7. The probability $P'r$(new boy hardworking) = 0.7 is specific to the new boy and is not related to the Pr(hardworking), which would be the proportion of boys in the school who are hardworking. Jeffrey's rule is similar to the theorem of total probabilities but with a mixture of population-estimated probabilities and specific beliefs.

In terms of the above example, Jeffrey's rule is

Pr\{(shoe_size of John large)\} =
Pr\{(shoe_size of John large)|(height of John tall)\}Pr\{(height of John tall)\}
+ Pr\{(shoe_size of John large)|¬(height of John tall)\}Pr\{(height of John tall)\}

We know

Pr\{(height of John tall)\} is contained in the interval $[x_1, x_2]$.

From this we can deduce

Pr\{(shoe_size of John large)\} is contained in the interval $[y, 1]$
where $y = 0.95x_1$

since we know

Pr\{(shoe_size of John large)|(height of John tall)\} $\in$ [0.95, 1]

and

Pr\{(shoe_size of John large)|¬(height of John tall)\} $\in$ [0, 1].

We must now convert this to a statement containing only a fuzzy set but no probabilities.

From the basic concept of a support pair, we can state

Pr\{(shoe_size of John large)\} = y
Pr\{(shoe_size of John ¬large)\} = 1 − 1 = 0
Pr\{(shoe_size of John any_possible_size)\} = $1 - y$

We use these three conclusions to determine a membership function for the fuzzy set f in the statement

$$(\text{shoe\_size of John } f)$$

by calculating f as the expected fuzzy set. Thus

$$\mu_f(s) = x\mu_{\text{large}}(s) + (1 - x) \quad \text{for all } s$$

We can defuzzify this fuzzy set, as described later. Briefly, we use the fuzzy set f to determine a least prejudiced probability distribution over the shoe_size domain and choose the size with the highest probability. If the domain for shoe_size had been a continuous domain, then we would defuzzify by choosing the mean of the distribution.

If point semantic unification is used rather than the interval semantic unification, then Fril would give the above solution but with $y = 0.95x$.

9.5.4 Fril Inference for a Single Rule

Consider the inference for a single Fril rule of the form

$$((h)((b_1))\ldots((b_n)))\text{: } ((u_1\ v_1)(u_n\ v_n))$$

when the following facts are given:

$$((b_i))\text{: } (\alpha_i\ \beta_i)\text{; all } i$$

More generally, the facts will not completely match the terms in the rule and the support pair $(\alpha_i\ \beta_i)$; and i will be determined using semantic unification. A generalized Jeffrey's rule for support pairs is the basic inference rule of Fril, as discussed above, so that h: $(z_1\ z_2)$ where

$$z_1 = \min \sum_i u_i\theta_i \text{ where } \alpha_i \le \theta_i \le \beta_i$$
$$\text{and } \sum_i \theta_i = 1$$
$$z_2 = \max \sum_i v_i\theta_i \text{ where } \alpha_i \le \theta_i \le \beta_i$$
$$\text{and } \sum_i \theta_i = 1$$

These are trivial optimization problems.

Each b_i can be a conjunction of terms, a disjunction, and a mixture of the two. A calculus based on probability theory is used to compute the support pair for any b_i with respect to the support pairs of its individual terms.

The inference rule for the basic rule is a special case of this, since the basic rule is equivalent to

$$((h)(((b))((\neg b)))) : ((u_1\ v_1)(u_2\ v_2))$$

For the evidential logic rule of the form

$$\begin{aligned}&((h))(\text{evlog}\ f\\&(c_1\ w_1)\ldots(c_n\ w_n)))\\&:(x_1\ y_1)(x_2\ y_2))\end{aligned}$$

with facts

$$((c_i)) : (\alpha_i\ \beta_i)$$

the support pair given to the body of the rule is

$$\left(\sum_i w_i\alpha_i \sum_i w_i\beta_i\right)$$

The basic inference rule is then used to give the final support pair for the head (h).

The point semantic unification case is only a special case of this where the supports (α_i, β_i) are replaced with point values.

9.5.5 Multiple Rule Case

More generally, Fril can use several rules with the same head predicate to determine a given inference. Consider, for example, the fuzzy logic rules

$$\begin{gathered}((y \text{ value is } f_1)\ (x_1 \text{ value is } g_1)\ (x_2 \text{ value is } h_1))\\((y \text{ value is } f_2)\ (x_1 \text{ value is } g_2)\ (x_2 \text{ value is } h_2))\\((y \text{ value is } f_n)\ (x_1 \text{ value is } g_n)\ (x_2 \text{ value is } h_n))\end{gathered}$$

for determining the value of y given values for x_1 and x_2. $\{f_i\}$, $\{g_i\}$, and $\{h_i\}$ are fuzzy sets defined on the domains for y, x_1, and x_2, respectively. If we provide the facts,

$$\begin{gathered}((x_1 \text{ is about\_}a))\\((x_2 \text{ is about\_}b))\end{gathered}$$

where about_a is a fuzzy set defined on the domain for x_1 and about_b a fuzzy set defined on the domain for x_2. Then Fril uses each rule to obtain

$$\begin{gathered}(y \text{ value is } f_1) : (x_1\, y_1)\\(y \text{ value is } f_2) : (x_2\, y_2)\\(y \text{ value is } f_n) : (x_n\, y_n)\end{gathered}$$

Fril then determines

$$(y \text{ value is } f_{a1})$$
$$(y \text{ value is } f_{a2})$$
$$(y \text{ value is } f_{an})$$

where f_{ak} is an expected fuzzy set determined as described previously. These are intersected to give the final solution

$$(y \text{ value is } f_a)$$

where $f_a = f_{a1} \cap f_{a2} \cap \ldots \cap f_{an}$ and $\cap$ is fuzzy intersection.

For multiple rules with the same head where the heads do not contain fuzzy sets, then the support pairs are intersected.

9.5.6 Interval and Point Semantic Unification

We will first explain the concepts involved in the Fril semantic unification using a simple example. This explanation will be in terms of discrete fuzzy sets. Fril handles both discrete and continuous fuzzy sets, and the algorithm is optimized for computational efficiency.

Consider the Fril program:

```
set (dice_dom (1, 2, 3, 4, 5, 6))
(small {1:1, 2:1, 3:0.3} dice_dom)
(about_2 {1:0.3, 2:1, 3:0.3} dice_dom)
((dice shows small))
```

If we ask the query

qs((dice shows about_2))

which asks for the support that the dice shows about_2, then Fril returns

((dice shows about_2)): (0.3 1)

The point semantic query

qs_p((dice shows about_2))

returns

((dice shows about_2)): 0.615

In other words, Fril calculates *Pr*{(dice shows about_2)|(dice shows small)} $\in$ [0.3, 1] for interval semantic unification and *Pr*{(dice shows about_2) | (dice shows small)} = 0.615 for point semantic unification. How is this done?

The fuzzy sets small and about_2 can be written as mass assignments [Baldwin 1992], namely,

$$m_{\text{small}} = \{1, 2\}: 0.7,\ \{1, 2, 3\}: 0.3$$
$$m_{\text{about\_2}} = \{2\}: 0.7,\ \{1, 2, 3\}: 0.3$$

where a mass assignment is equivalent in this case to a Dempster/Shafer basic probability assignment. We can depict these graphically as in the table below. The given information is depicted at the top of the table. In each cell we can denote the truth of the left-hand set given the top set. This truth value will be t, f, or u, representing true, false, or uncertain, respectively. For example, the truth of $\{2\}$ given $\{1, 2\}$ is uncertain since if the dice shows 1, then $\{2\}$ will be false, while if it shows 2, then $\{2\}$ will be true. What mass should we associate with each of the cells? Baldwin's theory of semantic unification states that the masses in the cells should satisfy the following row and column constraints: The column cell masses should sum to the column mass, and the row cell masses should sum to the corresponding row mass.

	0.7 $\{1, 2\}$	0.3 $\{1, 2, 3\}$
0.7 $\{2\}$	u $m11$	u $m12$
0.3 $\{1, 2, 3\}$	t $m21$	t $m22$

[0.3, 1]

Thus

$$m11 + m12 = 0.7$$
$$m21 + m22 = 0.3$$
$$m11 + m21 = 0.7$$
$$m12 + m22 = 0.3$$

This will not provide a unique solution. One solution is to multiply the column and row masses to obtain the corresponding cell mass. This procedure can be thought of as assuming independence of the mass assignment in the Fril program and of that given in the query. Fril uses this multiplication model, giving

$$m11 = 0.49,\ m12 = 0.21,\ m21 = 0.21, \text{ and } m22 = 0.09.$$

Thus we have the truth mass assignment

$$t\text{: } 0.3,\ \{t, f\}\text{: } 0.7$$

so that the support for *Pr*(about_2|small) = [0.3, 1].

A point semantic solution is obtained in the same way, but *m*11 and *m*12 are modified to give their contributions to true, assuming an equally likely probability distribution for dice values for the given information. Therefore we modify *m*11 to 0.5*m*11 and *m*12 to (1/3)*m*12, since {2} is true if 1 of {1, 2} is given and false otherwise, and {2} is true if 1 of {1, 2, 3} is true and false otherwise. This provides the modified table below:

	0.7 {1, 2}	0.3 {1, 2, 3}
0.7 {2}	0.245	0.07
0.3 {1, 2, 3}	0.21	0.21

0.615

If there are cells with an *f* entry, then the upper support for interval semantic unification will be less than 1.

The point semantic unification satisfies the normalization condition and the Dubois/Prade consistency condition, i.e.,

$$Pr(f|g) + Pr(f_c|g) = 1$$
$$Pr(A|g) \leq \Pi(A|g)$$

where f and g are fuzzy sets defined on the same domain, f_c is the complement of f, A is any subset of the domain, and Π is Zadeh's possibility measure. The multiplication model arises from relative entropy considerations discussed by Baldwin [1991], as does the use of Jeffrey's rule for inference. It should be noted that if the prior on the domain elements is different to equally likely distribution, then this will be taken into account when the point semantic unification is performed. Suppose in the above dice example it was known that the dice was weighted and had the prior {1: 1/9, 2:2/9, 3: 1/9, 4:2/9, 5: 1/9, 6:2/9}; then

$$Pr(\text{about\_2}|\text{small}) = (0.49)2/3 + (0.07)1/2 + 0.3 = 0.6617$$

9.5.7 *Least Prejudiced Distribution and Learning*

The fuzzy sets occurring in the various Fril rules can be determined automatically from a database of examples. For example, suppose we have a database of values of $y = F(x)$ for a range of values of x and we want to approximate the function using the fuzzy logic rules

$$((y \text{ has value in } f_i)(x \text{ has value in } g_i)) \text{ for } i = 1, \ldots, n$$

where $\{f_i\}$ and $\{g_i\}$ are fuzzy sets defined on the X and Y domains, respectively. Suppose further that we choose the $\{f_i\}$ to be triangular fuzzy sets on the Y domain. How should we choose $\{g_i\}$ to provide a good approximation to the function? The inference method for a given input for X is that described in sections 4 and 5. Defuzzification using the mean of the least prejudiced distribution is used as the estimate for $F(x)$.

In this section, we will define what is meant by the least prejudiced distribution, outline the method used to determine the fuzzy sets $\{g_i\}$, and indicate how this can be extended to the case of the evidential logic rule. The theory is described by Baldwin [1994].

Consider a discrete fuzzy set small for the dice problem above. The statement (dice score is small) provides a possibility distribution over the dice domain where $\pi(i) = \mu_{\text{small}}(i)$, $i = 1, \ldots, 6$.

According to Baldwin's theory of mass assignments, this is equivalent to a family of probability distributions given by the mass assignment

$$m_{\text{small}} = \{1, 2\}: 0.7, \{1, 2, 3\}: 0.3$$

The mass 0.7 can be distributed among the elements 1 and 2 in any way and the mass 0.3 among 1, 2, 3 in any way. This gives the family of probability distributions. The least prejudiced distribution is the one given by allocating a mass equally among the elements with which it is associated. Thus the least prejudiced distribution for the fuzzy set *small* is

$$\text{lpd}_{\text{small}} = 1:0.35 + 0.1, 2:0.35 + 0.1, 3:0.1$$

giving

$$\text{lpd}_{\text{small}} = 1:0.45, 2:0.45, 3:0.1$$

Fril extends this to the continuous case and provides a least prejudiced distribution for any fuzzy set.

Defuzzification instantiates the value to the mean value of this least prejudiced distribution.

Suppose we have a frequency distribution $f(x)$ for values of the attribute X determined from a set of examples. Fril determines the appropriate fuzzy set for F by ensuring that the least prejudiced distribution for this fuzzy set is f. If the classification is fuzzy, as in the above rules for function approximation, then Fril takes into account the fact that for some examples the classification will have a membership in several rule heads.

If we have a set of examples and for each example we are provided with attribute values for attributes $F_1, \ldots, F_n$ and a given classification (c, say), we can use the above method to derive the fuzzy sets occurring as feature values in the evidential logic rule. Fril can also determine near optimal weights for the rule using a specialized discrimination algorithm.

This approach has been used for function approximation; several classification-type problems, such as handwriting character recognition and underwater sound recognition from acoustic spectra; and deriving fuzzy control rules. The method is an alternative approach to neural supervised learning and can be used for similar types of problems.

9.5.8 Applications of Fril

The Fril language is an uncertainty logic programming system that can be used for fuzzy control, evidential logic reasoning, causal reasoning, classification, and other AI applications that require reasoning with missing information, vague information, or uncertain information.

It can be used to build expert systems, decision support systems, vision understanding systems, fuzzy databases, and other AI knowledge engineering applications [Baldwin and Martin 1993].

For example, Fril has been used to implement an intelligent data browser. A window-environment front end is provided that allows the user to enter a database or link to an existing database in Oracle, input rules, and ask any relevant queries concerning the database. The required evidential logic and other rules required to answer a particular query will automatically be constructed. The user can ask for an explanation and can investigate the sensitivity of any new rules formed. Queries can be asked about any attribute of the database when given information concerning other attributes of the database. The given information need not be precise and can be in the form of fuzzy sets or intervals or sets of values. The user can contribute to the establishment of the required rules in various ways—for example, choosing the type of rule, the features in the body of a rule, the weights in an evidential logic rule, or the fuzzy sets in a rule. These decisions can be made by the intelligent browser automatically, but the user can then make any changes if required. Rules formed are retained for future use. When appropriate, the accuracy of a new rule can be tested by using the database as test cases for which the answers are known.

This type of module has many applications from scientific, engineering, financial, and business fields. The system can be used to provide a summary of large amounts of data, interpolate between database instances, provide approximate reasoning, derive classifiers, perform case-based reasoning, derive causal nets, derive probabilistic fuzzy rules, and derive fuzzy controllers.

In the case of classification, for example, the classification could be the suitability of a house for a given customer and the features would be the various qualities of the house such as size of garden, number of bedrooms, size of lounge, etc. A representative number of examples of suitable houses would be chosen by the customer. A new house on the market could then be tested to see for which customers it would be suitable. The database could be the classification of creditworthiness of persons. The classification of creditworthiness could be {very_good, good, average, poor, very_poor}. The database would consist of past customers with their details as features and subjective creditworthiness estimated. Another example might be a classification of change in interest rate with features representing economic measurable conditions. Classes of {very_good, good, average, poor, very_poor} for the potential for oil at a given place with geological measurement and other features is another obvious example.

Fril has been successfully used to build an expert system for designing aircraft structures using composite materials. This expert system calls various analysis programs in different languages to help with the design and evaluation. Fril has also been used for command and control studies, a dental expert system for planning orthodontic treatment, design of a client administration expert system, to produce a modeling tool for representing the behavior of aircrew in aircrew and fixe wing operations, to build an intelligent manual for safety studies in the disposal of nuclear waste, software dependability studies, and conceptual graph implementation.

Exercises

1. Consider the linguistic variable "Age." Let the term "old" be defined by

$$\mu_{\text{old}}(x) = \begin{cases} 0 & \text{if} \quad x \in [0, 40] \\ \left(1 + \left(\dfrac{x-40}{5}\right)^{-2}\right)^{-1} & \text{if} \quad x \in (40, 100] \end{cases}$$

   Determine the membership functions of the terms "very old," "not very old," "more or less old."

2. Let the term "true" of the linguistic variable "Truth" be characterized by the membership function

$$T(\nu;\alpha,\beta,\gamma)=\begin{cases}0 & \text{if} \quad \nu\le\alpha \\ 2\left(\dfrac{\nu-\alpha}{\gamma-\alpha}\right)^2 & \text{if} \quad \alpha\le\nu\le\beta \\ 1-2\left(\dfrac{\nu-\gamma}{\gamma-\alpha}\right)^2 & \text{if} \quad \beta\le\nu\le\gamma \\ 1 & \text{if} \quad \nu\le\gamma\end{cases}$$

Draw the membership function of "true." Determine the membership functions of "rather true" and "very true." What is the membership function of "false" = not "true" and what of "very false"?

3. What is the essential difference between Baldwin's definition of "true" and Zadeh's definition?
4. Let the primary terms "young" and "old" be defined as in example 9–3. Determine the secondary terms "young and old," "very young," and "not very old."
5. Let "true" and "false" be defined as in example 9–4. Find the membership function of "very very true." Compare the fuzzy sets "false" and "not true."
6. Let the universe $X = \{1, 2, 3, 4, 5\}$ and "small integers" be defined as $\tilde{A} = \{(1, 1), (2, .5), (3, .4), (4, .2)\}$. Let the fuzzy relation "almost equal" be defined as follows:

$\tilde{R}$:

	1	2	3	4
1	1	.8	0	0
2	.8	1	.8	0
3	0	.8	1	.8
4	0	0	.8	1

What is the membership function of the fuzzy set B = "rather small integers" if it is interpreted as the composition $\tilde{A} \circ \tilde{R}$?

7. What is the relationship between a relational assignment equation and a possibility assignment equation?
8. Which of the definitions of "true" amounts to unity possibility distributions and which other important linguistic variables are represented by unity possibility distribution?
9. Consider examples 9–10 and make propositions about cars like Mercedes, Volvo, Chevy, and Rolls Royce.

10 FUZZY SETS AND EXPERT SYSTEMS

10.1 Introduction to Expert Systems

During the last three decades, the potential of electronic data processing (EDP) has been used to an increasing degree to support human decision making in different ways. In the 1960s, the management information systems (MISs) created probably exaggerated hopes for managers. Since the late 1970s and early 1980s, decision support systems (DSSs) found their way into management and engineering. The youngest offspring of these developments are the so-called knowledge-based expert systems or short expert systems, which have been applied since the mid-1980s to solve management problems [Zimmermann 1987, p. 310]. It is generally assumed that expert systems will increasingly influence decision-making processes in business in the future.

If one interprets decisions rather generally, that is, including evaluation, diagnosis, prediction, etc., then all three types could be classified as decision support systems that differ gradually with respect to the following properties:

1. Does the system "optimize" or just provide information?
2. Is it usable generally or just for specific purposes and areas?

3. Is it self-contained with respect to procedures and algorithms, or does it "learn" and "derive" inference and decision-making rules from knowledge that is inquired from a human (expert) and analyzed within the system?

It can be expected that in the future these decision support systems will contain to an increasing degree features of all three types of the above-mentioned systems. Even though fuzzy set theory can be used in all three "prototypes," we shall concentrate on "expert systems" only because the need and problem of managing uncertainty of many kinds is most apparent there; and hence the application of fuzzy set theory is most promising and, in fact, most advanced. In operations research (OR), the modeling of problems is normally being done by the OR specialist. The user then provides input data, and the mathematical model provides the solution to the problem by means of algorithms selected by the OR specialist.

In expert systems, the domain knowledge is typically emphasized over formal reasoning methods:

> In attempting to match the performance of human experts, the key to solving the problem often lies more in specific knowledge of how to use the relevant facts than in generating a solution from some general logical principles. "Human experts achieve outstanding performance because they are knowledgeable" [Kastner and Hong 1984].

Conventional software engineering is based on procedural programming languages. The tasks to be programmed have to be well understood, the global flow of the procedure has to be determined, and the algorithmic details of each subtask have to be known before actual programming may proceed. Debugging often represents a huge investment of time, and there is little hope of automatically explaining how the results are derived. Later modification or improvement of a program becomes very difficult.

> Most of the human activities concerning planning, designing, analyzing, or consulting have not been considered practical for being programmed in conventional software. Such tasks require processing of symbols and meanings rather than numbers. But more importantly, it is extremely difficult to describe such tasks as a step-by-step process. When asked, an expert usually cannot procedurally describe the entire process of problem solving. However, an expert can state a general number of pieces of knowledge, without a coherent global sequence, under persistent and trained interrogation. Early AI research concentrated on how one processes relevant relations that hold true in a specific domain to solve a given problem. Important foundations have been developed that enable, in principle, any and all logical consequences to be generated from a given set of declared facts. Such general purpose problem solving techniques, however, usually become impractical as the toy world used for demonstration is

replaced by even a simple real one. The realization that knowledge of how to solve problems in the specific domain should be a part of the basis from which inferences are drawn contributed heavily to making expert systems technology practical [Kastner and Hong 1984].

While the typical OR model or software package normally supports the expert, an expert system is supposed to model an expert and make his or her expert knowledge available to nonexperts for purposes of decision making, consulting, diagnosis, learning, or research.

The character of an expert system might become more apparent if we quote some of the system characteristics considered to be attributes of expert systems [Konopasek and Jayaraman 1984]. Attributes of expert systems include:

The expert system has separate domain-specific knowledge and problem-solving methodology and includes the concepts of the knowledge base and the inference engine.

The expert system should think the way the human expert does.

Its dynamic knowledge base should be expandable and modifiable and should facilitate "plugging in" different knowledge modules.

The interactive knowledge transfer should minimize the time needed to transfer the expert's knowledge to the knowledge base.

The expert system should interact with the language "natural" to the domain expert; it should allow the user to think in problem-oriented terms. The system should adapt to the user and not the other way around. The user should be insulated from the details of the implementation.

The principal bottleneck in the transfer of expertise—the knowledge engineer—should be eliminated.

The control strategy should be simple and user-transparent; the user should be able to understand and predict the effect of adding new items to the knowledge base. At the same time, the strategy should be powerful enough to solve complex problems.

There should be an inexpensive framework for building and experimenting with expert systems.

The expert system should be able to reason under conditions of uncertainty and insufficient information and should be capable of probabilistic reasoning.

An expert system should be able to explain "why" a fact is needed to complete the line of reasoning and "how" a conclusion was arrived at.

Expert systems should be capable of learning from experience.

Cutting a long story short, Kastner and Hong [1984] provide this definition:

> *An expert system* is a computer program that solves problems that heretofore required significant human expertise by using explicitly represented domain knowledge and computational decision procedures [Kastner and Hong 1984].

A sample of some other definitions of an expert system can be found in the work of Fordyce et al. [1989, p. 66]. The general structure of an expert system is shown in figure 10–1 (see also Zimmermann [1987, p. 262]). In the following, the five components of such a system are explained in more detail. The *knowledge acquisition module* supports the building of an expert system's knowledge base.

> The subject of knowledge acquisition for knowledge-based systems falls conveniently into two parts depending on whether the knowledge is elicited from the experts by knowledge engineers or whether that knowledge is acquired automatically by the computer using some form of automatic learning strategy and algorithms [Graham and Jones 1988, p. 279].

A module that aids the knowledge engineer during the process of knowledge elicitation could consist of a user-friendly rule editor, an "automatic error-checking when rules are being put in, and good online help facilities" [Ford 1987, p. 162]. (See also Buchanan et al. [1983, p. 129]). AQUINAS is such a system; it is presented by Boose [1989, p. 7].

Another way to acquire domain-dependent knowledge is the application of machine learning techniques to automatically generate a part of the knowledge base. It is expected that rapid improvements will take place in the field of automatic knowledge acquisition in the future. The interested reader is referred to Michalski et al. [1986, p. 3] and Morik [1989, p. 107].

The *knowledge base* contains all the knowledge about a certain domain that has been entered via the above-mentioned knowledge acquisition module. Apart from special storage requirements and system-dependent structures, the knowledge base can be exchanged in some expert systems. That means that there can be several knowledge bases, each covering a different domain, which can be "plugged into" the "shell" of the remaining expert system.

> There are basically two types of knowledge that will need to be represented in the system; *declarative knowledge* and *procedural knowledge*. The declarative part of the knowledge base describes "what" the objects (facts, terms, concepts, . . .) are that are used by the expert (and the expert system). It also describes the relationships between these objects. This part of the knowledge base is sometimes referred to as the "data base" or "facts base."

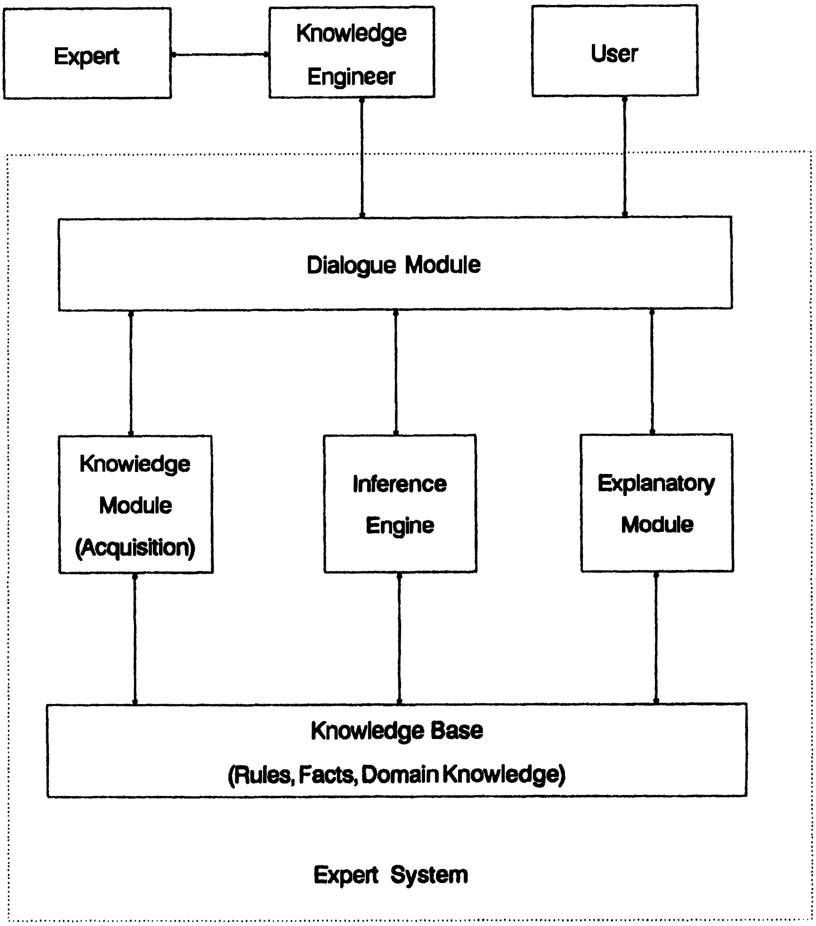


Figure 10–1. Structure of an expert system.

> The procedural part of the knowledge base contains information on how these objects can be used to infer new conclusions and ultimately arrive at a solution. Since this "how-to" knowledge is usually expressed as (heuristic or other) rules, it is generally known as the rule-base [Rijckaert et al. 1988, p. 493].

A number of techniques for representing the expert knowledge have been developed. These are described by Barr and Feigenbaum [1981/82] in greater detail. The four methods most frequently used in expert systems are production rules, semantic nets, frames, and predicate calculus (see Zimmermann [1987,

p. 266]). While we will investigate here the first three of these, the reader is referred to Nilsson [1980, p. 132] for the latter.

Production Rules. Production rules are by far the most frequently used method for representing procedural knowledge in an expert system. They are usually of the form: "If a set of conditions is satisfied, then a set of consequences can be produced."

> Production rules are used to capture the expert's rule of thumb or heuristic as well as useful relations among the facts in the domain. These if-then rules provide the bulk of the domain-dependent knowledge in rule-based expert systems and a separate control strategy is used to manipulate the rules.
>
> If the car won't start and
> the car lights are dim
> then the battery may be dead.
>
> Many experts have found rules a convenient way to express their domain knowledge. Also, rule bases are easily augmented by simple adding more rules. The ability to incrementally develop an expert system's expertise is a major advantage of rule-based schemes [Kastner and Hong 1984].

Semantic Nets. One method of encoding declarative knowledge is a semantic net. Concepts, categories, or phenomena are presented by a number of nodes associated with one another by links (edges). These links may represent causation, similarity, propositional assertions, and the like. On the basis of these networks, insight into structures can be gained, inferences can be made, and classifications can be obtained. In figure 10–2, a semantic net is used to represent declarative knowledge about the structure of some vehicles.

Frames. The concept of a frame for representing knowledge in an expert system is introduced by Minsky [1975]: "A frame is a structure that collects together knowledge about a particular concept and provides expectations and default knowledge about that concept." Typically, the frame is represented in the computer as a group of slots and associated values. The values may themselves be other frames.

Frame:	vehicle
classes	passenger, motorcycle, truck, bus, bicycle, . . .
wheels	(integer)
propelled by	motor, human feet. . . .

.
.
.

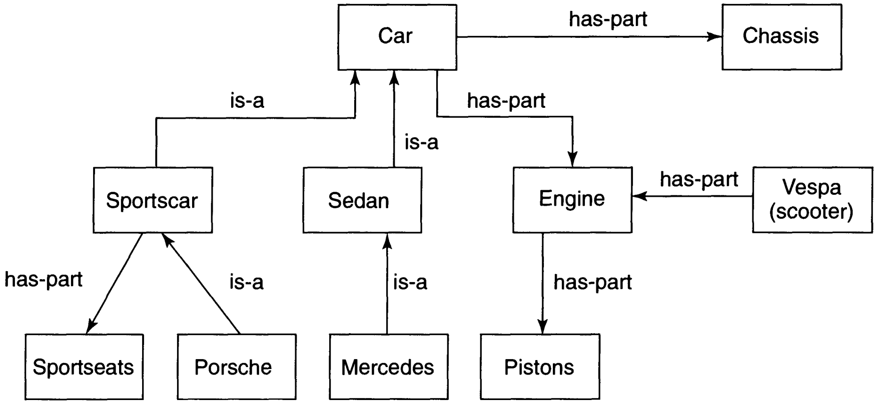


Figure 10–2. Semantic net.

Frame:	bicycle
is-a	vehicle
wheels	2 (default)
capacity	1 person (default)

.
.
.

[Kastner and Hong 1984]

New concepts can often be represented by adding frames or by putting new information in "slots" of existing frames. Slots in frames may also be used for inference rules and empty slots might indicate missing information.

> The inclusion of procedures in frames joins together in a single representational strategy two complementary (and, historically, competing) ways to state and store facts: procedural and declarative representations [Harmon and King 1985, p. 44].

The *inference engine* is a mechanism for manipulating the encoded knowledge from the knowledge base and to form inferences and draw conclusions. The conclusions can be deduced in a number of ways that depend on the structure of the engine and the method used to represent the knowledge. In the case of production rules for knowledge encoding, different control strategies have been used that direct input and output and select which rules to evaluate. Two very popular strategies are "forward chaining" and "backward chaining." In the former, data-driven rules are evaluated for which the conditional parts are satisfied. The latter strategy (goal-driven) selects a special rule for evaluation. The

Table 10–1. Expert systems.

Name	*Domain of expertise*	*Major technique*
CADIAG-2 [Adlassnig et al. 1985]	internal medicine	rules*
DENDRAL [Lindsay et al. 1980]	molecular structure elucidation	rules
EMERGE [Hudson and Cohen 1988]	chest pain analysis	rules*
ESP [Zimmermann 1989]	strategic planning	rules*
EXPERT [Weiss and Kulikowski 1981]	rheumatology, ophthalmology	rules* hierarchies
FAULT [Whalen et al. 1987]	financial accounting	rules*
MYCIN [Buchanan and Shortliffe 1984]	infectious disease diagnosis and treatment	rules
OPAL [Bensana et al. 1988]	job shop scheduling	rules*
PROSPECTOR [Benson 1986]	mineral exploration	inference network
R1/XCON [McDermott 1982]	computer configuration	rules
SPERIL [Ishizuka et al. 1982]	earthquake engineering	rules*

* Includes fuzzy logic.

"goal" is to satisfy the conditional part of this rule. If this cannot be achieved directly, then subgoals are established on the basis of which a chain of rules can be established such that eventually the conditional part of the first rule can be satisfied. Further information about inference strategies has been described by Waterman [1986].

The above-mentioned approaches can, of course, be combined. In addition to these techniques, expert systems may also contain rather sophisticated mathematical algorithms, such as cluster algorithms and optimization and search techniques like tabu search (see Glover and Greenberg [1989, p. 119]). This development is actually already in the direction of decision support systems, but in many cases it will make the expert system more efficient and even more user-friendly. Table 10–1 gives some indication in which area expert systems are already available and what techniques they use. By no means does this table claim to be exhaustive.

10.2 Uncertainty Modeling in Expert Systems

There are three main reasons for the use fuzzy set theory in expert systems:

1. The interfaces of the expert system on the expert side as well as on the user side are with human beings. Therefore communication in a "natural" way seems to be the most appropriate; and "natural" means, generally, in the language of the expert or user. This suggests the use of linguistic variables as they were described in chapter 9.
2. The knowledge base of an expert system is a repository of human knowledge, and since much of human knowledge is imprecise in nature, it is usually the case that the knowledge base of an expert system is a collection of rules and facts that, for the most part, are neither totally certain nor totally consistent [Zadeh 1983a, p. 200]. The storage of this vague and uncertain portion of the knowledge by using fuzzy sets seems much more appropriate than the use of crisp concepts and symbolism.
3. As a consequence of what has been said in point 2, the "management of uncertainty" plays a particularly important role. Uncertainty of information in the knowledge base induces uncertainty in the conclusions, and therefore the inference engine has to be equipped with computational capabilities to analyze the transmission of uncertainty from the premises to the conclusions and to associate the conclusion with some measure of uncertainty that is understandable and properly interpretable by the user. The reader should also recall from chapter 1 that imprecision in human thinking and communication is often a consequence of abundance of information, that is, the fact that humans can often process the required amount of information efficiently only by using aggregated (generic) information. This efficiency of human thinking, when modeled in expert systems, might also increase efficiency, that is, decrease answering time and so on.

Most of the expert systems existing so far contain an inference engine on the basis of dual logic. The uncertainty is taken care of by Bayesian probability theory. The conclusions are normally associated with a certainty or uncertainty factor expressing stochastic uncertainty, confidence, likelihood, evidence, or belief. Only recently have the designers of expert systems become aware of the fact that all of the types of uncertainty mentioned above cannot be treated the same way and that a factor of, for example, .8 to express the uncertainty of a conclusion does not mean very much to the user. The expert systems marked with an asterisk in table 10–1 are already using fuzzy set approaches in different ways. We shall illustrate some of them later. In addition, proposals have been published on how fuzzy set theory could be used meaningfully in expert systems.

The most relevant approaches in fuzzy set theory are fuzzy logic and approximate reasoning for the inference engine [Lesmo et al. 1982; Sanchez 1979]; the presentation of conditions, indicators, or symptoms by fuzzy sets, especially linguistic variables, to arrive at judgements about secondary phenomena [Esogbue and Elder 1979; Moon et al. 1977; etc.]; the use of fuzzy clustering for diagnosis [Fordon and Bezdek 1979; Esogbue and Elder 1983]; and combinations of fuzzy set theory with other approaches, for example, Dempster's theory of evidence [Ishizuka et al. 1982], to obtain justifiable and interpretable measures of uncertainty.

In chapter 9 we have already discussed fuzzy logic and its relationship to classical dual logic. Here we shall additionally focus on the if-then relationship, which is generally assumed to be deterministic. If this is not the case, we have to "qualify" its character.

We shall distinguish three kinds of qualifications: *truth qualification*, *probability qualification*, and *possibility qualification*. Qualifications of statements are possible or even necessary, independent of whether the statement or phenomenon is crisp or fuzzy. The kind of modeling, however, will have to be different.

There is a difference between the *truth* of a part of a statement, a fact, or an antecedent and the *truth* of a compound statement. While the former depends on the antecedent's conformity or compatibility with reality, the latter depends, in addition, on the type of connectives used to build the compound statement from its parts. We will discuss the former under "matching"; the latter will be considered when discussing uncertainty in the process of inference. The reader is referred to the first part of this chapter with respect to truth qualification in fuzzy logic and approximate reasoning, and also to the section about possibility qualification further on.

Probability Qualification. It is not surprising that probability qualifications are still the most common way to characterize uncertainty with respect to the occurrence of an event (which might be the real occurrence of the predicted—"true"—outcome of a conclusion). Probability theory has long been the only way to model uncertainty and therefore, is still the most accepted method. Of course, probability has often been abused to model all kinds of uncertainty!

In the following we shall briefly discuss probability qualifications as point estimates, intervals, and (possibility) distributions. These approaches assume crisply defined events. For models of the probability of fuzzy events, the reader is referred to chapter 8 of this book [Dubois and Prade 1980a, pp. 141–144; Yager 1984, pp. 273–283].

Let us consider the rule

If A then C	
A is true	(antecedent)
Then C is true	(conclusion)

In the most frequently applied Bayesian approach, the Bayes inversion theorem is used:

$$\Pr(C/A) = \frac{\Pr(C)}{\Pr(A)} \Pr(A/C) \tag{10.1}$$

Hence, $\Pr(C/A)$ is the probability of C given A, $\Pr(C)$ the probability of C, etc. If the antecedent has the possible states A_i and the conclusion has the possible states C_j, then (10.1) becomes

$$\Pr(C_j/A_i) = \frac{\Pr(C_j)}{\Pr(A_i)} \Pr(A_i/C_j) \tag{10.2}$$

(Determination of probabilities of conclusions in larger inference systems shall not be discussed here, because textbooks on probability theory exist in abundance.)

Objections against this approach are, first of all, that aspects of uncertainty that are nonprobabilistic in nature may be included. Computationally this approach becomes prohibitive if the events (antecedent, conclusion) are considered to be fuzzy—represented as fuzzy sets. A second criticism is the need to identify point values for the probabilities of events that may by far be overstatements of our actual knowledge of the likelihood of occurrence of that particular event.

The criticism has lead Dempster [1967] to suggest the concept of upper and lower probabilities and Shafer [1976] to present his *theory of evidence*. The basic concept of this theory is that instead of representing the probability of an event A by a point value, $\Pr(A)$, it may be bounded by the subinterval $[\underline{\Pr}(A), \overline{\Pr}(A)]$ of $[0,1]$. This theory has some connections to the theory of fuzzy sets and shall, therefore, be discussed in some more detail. Rather than following a purely probabilistic line of argument, see e.g. [Dubois and Prade 1982, p. 171; Goodman and Nguyen 1985] we shall follow Zadeh's line of argument [Zadeh 1984], which seems easier to comprehend and closer to "fuzzy thinking". After an introduction to the basic ideas of Dempster and Shafer, we will return to the more common representation of their theory.

Let us consider the following introductionary example:

Table 10–2. A crisp data base.

Emp 1	*Name*	*No. of children*
	1	1
	2	3
	3	5
	4	2
	5	4

Table 10–3. An extended data base.

Emp 2	*Name*	*No. of children*	*Between 3 and 5 children?*
	1	1,2	impossible
	2	1	impossible
	3	4,5	certain
	4	5,6	possible
	5	6	impossible

Example 10–1

Let us assume we have a data base in which the (atomic) elements are related to each other by first-order relations. One of these may be as shown in table 10–2. In a simple range query of the type "what portion of the employees in the data base have between 1 and 3 children?" we would get, from table 10–2, the answer 3/5, which may be interpreted as the probability of an employee (contained in the data base) having between 1 and 3 children.

Let us now assume that our knowledge is less precise and that we only know the second-order relation shown in table 10–3. We now put the query: "What portion of the employees has between 3 and 5 children?". This is obviously possible for employees 3 and 4. It is not possible for employees 1, 2, and 5! Therefore, the statement "he has between 4 and 5 children" is certainly true for employee 3; it is possibly true for employee 4; and it is certainly not true for employees 1, 2, and 5.

In the Dempster-Shafer theory the portion of the intervals for which the statement is certainly true is called *lower probability*. In our example this is 1/5. As the *upper probability* they consider the portion of the elements (intervals) for which the statement can (possibly) be true (i.e. 1 minus the portion for which the statement cannot be true). In example 10–1 this is $(1 - 3/5 = 2/5)$.

The lower probability is also called *measure of belief* and the upper probability is called *measure of plausibility*. It should be noted that in our example the employees were considered as atomic elements (all equal probabilities!). If this is not the case, the different probabilities of the intervals will have to be taken into consideration when determining lower and upper probabilities. Shafer calls the sets of attributes (number of children) assigned to the elements focal elements and their probabilities of occurrence *basic probability assignment*. In example 10–1 the answer to the question "what is the probability of an employee having between 3 and 5 children?" would be: the lower probability (degree of belief) is 1/5 and the upper probability (plausibility) (degree of belief) is 2/5.

Example 10–1 was a rather intuitive example. Let us now define the uncertainty measures of the theory of evidence properly.

Definition 10–1 [Dubois and Prade 1982a, 1985b; Prade 1985; Goodman and Nguyen 1985, p. 32]

Let X be a finite set equipped with a probability measure Pr defined on the set P(X) of subsets of X. Consider a point-to-set mapping Γ from X to some set S. That is, $\forall_x \in X, \Gamma(x)$ is a subset of S. Let $f \subseteq S$ (f = focal element) and the mapping m from P(S) to [0,1] (basic probability assignment) be defined as follows:

$$\mathrm{m}(\varnothing) = 0$$

$$m(f) = \frac{\Pr(\{x \in X, \Gamma(x) = f\})}{1 - \Pr(\{x \in X, \Gamma(x) = \varnothing\})} \quad \forall f \subseteq S, f \neq \varnothing$$

Then the *upper probability* or *plausibility measure* is defined as

$$\Pr{}^*(Q) = PL(Q) = \sum_{f \cap Q \neq \varnothing} m(f) \tag{10.3}$$

The *lower probability*, *belief function*, or *credibility measure* (Dubois and Prade) is defined as

$$\Pr{}^*(Q) = Bel(Q) = Cr(Q) \sum_{f \subseteq Q} m(f) \tag{10.4}$$

In analogy to these measures of uncertainty, doubt or commonality measures and disbelief or incredibility measures have been defined [Goodman and Nguyen 1985, p. 321].

Remark: Plausibility and belief are, of course, not unrelated. The following properties hold:

$$PL(Q) = Bel(Q) = 1 \tag{10.5}$$

$$PL(Q) = Bel(Q) = 0 \tag{10.6}$$

$$PL(Q) = 1 - Bel(\neg Q) \tag{10.7}$$

$$PL(A \cap B) \leq PL(A) + PL(B) - PL(A \cup B) \tag{10.8}$$

$$Bel(A \cup B) \leq Bel(A) + Bel(B) - Bel(A \cap B) \tag{10.9}$$

(10–5) relates to the normalization condition

$$\sum_{f \in F} m(f) = 1 \tag{10.10}$$

which may lead to some problems [Zadeh 1984, pp. 6–10].

While Bel (Q) obviously considers evidence supporting Q, PL(Q) focuses on the evidence supporting the contrary. If F contains only singletons, then PL(Q) = Bel(Q); that is, these measures reduce to normal probabilities. So far we have looked at scalar measures (probabilities) and interval measures (belief, plausibility). If we consider probability as a linguistic variable, then a measure for the probability of an event is a term of the linguistic variable "probability"—a fuzzy set characterized by its membership function. The notions of plausibility and belief have also been extended from crisp event (as considered here) to fuzzy event. The reader is referred to [Dubois and Prade 1985a, p. 553; Smets 1981].

Possibility Qualification. We now return to example 10–1 and assume that in table 10–3 the number of children of the various employees are described by possibility distributions, see e.g. [Zadeh 1983b].

To review, a possibility distribution can formally be described by a fuzzy set. One difference between a possibility distribution and a fuzzy set, however, is that in a fuzzy set the elements of the support belong to the fuzzy set to various degrees while in a possibility distribution the possibilities indicate the degree of possibility with which a variable can adopt various values. A discrete possibility distribution shall be denoted by

$$\Pi = \{(x_i, \Pi_i)\}$$

Then 10–3 and 10–4 respectively, satisfy the following axioms [Shafer 1976]:

$$PL(A \cup B) = \max\{PL(A), PL(B)\} \tag{10.11}$$

$$Bel(A \cap B) = \min\{Bel(A), Bel(B)\} \tag{10.12}$$

A plausibility measure which satisfies (10–11) is called a *possibility measure* (Π), and a belief measure which satisfies (10–12) is called a *necessity measure* (N) [Prade 1985; Zadeh 1984]. (The latter is called a "consonant belief function" by

Shafer.) In contrast to (10–5) through (10–10), possibility measures (Π) and necessity measures (N) have the following properties:

$$\min\{N(Q), N(\neg Q)\} = 0 \tag{10.13}$$

$$\max\{\Pi(Q), \Pi(\neg Q)\} = 1 \tag{10.14}$$

$$\Pi(Q) < 1 \Rightarrow N(Q) = 0 \tag{10.15}$$

$$N(Q) > 0 \Rightarrow \Pi(Q) = 1 \tag{10.16}$$

Example 10–2

Let us now assume that the information available concerning the number of children of our employees is not as in table 10–3, but as in table 10–4. Let us now ask "how possible is it that an employee has 3 or 4 children?".

If we consider the possibility of 3 or 4 children as

$$\Pi = \max_{Q \cap f \neq \varnothing} (\Pi_i) = \max\{.6\} = .6$$

the necessity as

$$N = \max_{Q \cap f = \varnothing} (1 - \Pi_i) = \min\{.2,0,0,0,.2,0,0\} = 0$$

then our answer would have to be:
"The possibility of an employee having 3 to 4 children is .6, the necessity is 0."
It should be noted that other interpretations and definitions of "necessity" and "possibility" measures exist, see e.g. [Dubois and Prade 1985a; Prade 1985].

Quantification. In human communication and therefore, also in knowledge transfer, statements include quantifiers other than the two quantifiers available in dual logic or classical mathematics. Often these quantifiers are implicit rather than explicit. An assertion of the type "Frenchmen are very charming" often

Table 10–4. A possibilistic data base.

Emp 3	*Name*	*Poss. of having x children*
	1	{(1,.8),(2,1)}
	2	{(1,1)}
	3	{(4,.6),(5,1)}
	4	{(5,.8),(6,1)}
	5	{(6,1)}

really means "most (or almost all) Frenchmen are charming". Likewise the proposition "Hans is never late" would normally be interpreted as "Hans is late very rarely".

To model this and other types of quantifiers, fuzzy set theory includes *fuzzy quantifiers*. We shall view a fuzzy quantifier as "a fuzzy number which provides a fuzzy characterization of the absolute or relative cardinality of one or more fuzzy or nonfuzzy sets" [Zadeh 1982, p. 5]. Zadeh distinguishes between fuzzy quantifiers of the first kind (referring to absolute counts), and quantifiers of the second kind (referring to relative counts). Examples of the former are: several, few, many, etc. Examples of the latter kind are most, many, often, a large fraction, etc. Quantifiers of the third kind are ratios of quantifiers of the second kind (see also in chapter 9).

Scalar quantifiers are normally modeled using their cardinality or sigma count Let us consider the proposition "Vickie has several close friends" [Zadeh 1982, p. 11]. The fuzzy set "close friends of Vickie" may be represented by

$$\tilde{F} = \{(\text{Enrique, } 1), (\text{Ramon, } .8), (\text{Elie, } .7), (\text{Sergei, } .9), (\text{Ron, } .7)\}$$

Then the sigma count (cardinality) of

$$\tilde{F} = (1 + .8 + .7 + .8 + .7) = 4$$

If "several" plays the role of a specified subset of integers 1, . . . , 10, in which 4 is assumed to be compatible with the meaning of "several" to the degree .8, the above proposition may be modeled as

$$\text{Poss}\{\text{Count(close friends(Vickie))} = 4\} = .8$$

In some cases it might not be appropriate or desirable to express the cardinality of a fuzzy set as a number, rather as a fuzzy set. Zadeh proposed three notions of fuzzy counts based on the concept of α-level cuts:

Definition 10–2 [Zadeh 1982, p. 15]

Let $\tilde{F}$ be a (discrete) fuzzy set and F_α an α-level cut of fuzzy set $\tilde{F}$. Card_α represents the cardinality (count) of the elements of an α-level cut.

The *FG*-count is then defined to be the fuzzy set

$$FG = (Card_{\alpha_i}, \sup \alpha\{\alpha \mid Card_\alpha \geq i\}) \quad \text{i} = 0, \ldots, \text{n}$$

The *FL*-count is defined as

$$FL = \{(Card_\alpha, \sup_\alpha\{\alpha\}Card_\alpha \geq n - i\}) \quad \text{i} = 1, \ldots, \text{n}\}$$

The *FE*-count is the fuzzy set

$$FE = \{(Card_\alpha, \min\{\mu_{FG}(\alpha_i), \mu_{FL}(\alpha_i)\}) \quad \mathrm{i} = 1, \ldots, \mathrm{n}\}$$

The counts of definition 10–2 may be interpreted as follows: The FG-count is the truth value of the proposition "$\tilde{F}$ contains at least i elements", FL the truth of "$\tilde{F}$ contains at most i elements" and the FE-count of "$\tilde{F}$ contains exactly i elements".

Example 10–3 [Zadeh 1982, pp. 15–16]

Let

$$\tilde{F} = \{(X_1, .6), (x_2, .9), (x_3, 1), (x_4, .7), (x_5, .3)\}$$

The α-level sets are listed in table 10–5. The various counts are

$$\begin{aligned}
\mathrm{FG}(\tilde{F}) &= \{(0,1), (1,1), (2,.9), (3,.7), (4,.6), (5,.3)\}\\
\mathrm{FL}(\tilde{F}) &= \{[(2,.1), (3,.3), (4,.4), (5,.7), (6,1)] - 1\}\\
&= \{(2,.1), (2,.3), (3,.4), (4,.7), (5,1)\}\\
\mathrm{FE}(\tilde{F}) &= \{(1,.1), (2,.3), (3,.4), (4,.6), (5,.3)\}
\end{aligned}$$

The normal sigma count would be

$$|\tilde{F}| = \sum count(\tilde{F}) = \sum_i \mu_i(\alpha) = 3.5$$

Matching. By *matching problem* we mean the approximation of real evidence by assumed structures or of computational results by communication languages. In expert systems this problem occurs twice; whenever knowledge (relations between facts) contained in the knowledge base has to be used on the basis of observed facts that do not quite coincide with the "models of facts" in the knowledge base, or when it cannot be decided whether it coincides or not.

Table 10–5. α-level sets.

α	F_I
1.	$\{x_3\}$
.9	$\{x_2, x_3\}$
.7	$\{x_2, x_3, x_4\}$
.6	$\{x_1, x_2, x_3, x_4\}$
.3	$\{x_1, x_2, x_3, x_4, x_5\}$

The first case is represented by example 9–7 in which the knowledge base contains only the "fact" red tomatoes while the observed fact is "very red tomato". For the second case, consider the rule "if the rod is hot, stop the heating process". The observed fact could be "the rod has a temperature of 150°C". The question then is "is that rod hot or not hot?".

Let us call these two types of problems *input matching* and discuss methods for their solution further down. Another matching problem occurs when the result of the inference process has been obtained—e.g., as the membership function of a fuzzy set. The user of the system, however, does not want the answer as a function but in a language close to his own. The problem is then to search for a term of a linguistic variable whose membership function is very close to the one obtained by the system. This is, of course, a problem of output interpretation and we shall call it *output matching*.

The input-matching problem is obviously already reduced if the knowledge base contains descriptions in the form of fuzzy sets rather than only crisp models. Also, it has been suggested that in addition to using similarity relations, truth and certainty values be used to model the degree of compatibility of reality and model and to introduce it into the inference process. Another promising approach is the suggestion by Cayrol, Farrency, and Prace [1982] to use pattern matching where possibility measures and necessity measures are employed, in order to evaluate the semantic similarities between patterns (models) and data.

Output matching is more a psycholinguistic problem. It occurs primarily if approximate or plausible reasoning methods or other fuzzy approaches are used in which membership functions (of linguistic variables, for instance) are used. Even if at the input level the semantic meaning of data and formal knowledge representation coincides satisfactorily, the process of inference may yield membership functions that do not fit the membership functions of linguistic variables or their terms, as defined beforehand, well enough to communicate the results effectively to the user of an expert system.

Certainty factors or degrees of truth do not relay a missing correspondence well enough. Another approach, which seems to be promising but not yet well enough developed to be used efficiently, is the linguistic approximation mentioned in example 9–6.

We shall describe some more recent attempts to apply fuzzy set theory to knowledge representation and inference mechanisms in expert systems.

Although, in a precise environment, production rules are adequate to represent procedural knowledge (as was seen in section 10.1), this is no longer true in a fuzzy environment. One way to deal with imprecision is to use *fuzzy production rules*, where the conditional part and/or the conclusions part contains linguistic variables (see chapter 9). An application of this knowledge-representation technique in the area of job-shop scheduling has been given by Dubois [1989, p. 83]. Negoita [1985, p. 80] gives a basic introduction into fuzzy production rules.

While little work has been done in the field of "fuzzy semantic nets," suggestions to fuzzify frames to represent uncertain declarative knowledge, and an illustrative example, stem from Graham and Jones [1988, p. 67]. The two main generalizations for arriving at a *fuzzy frame* are

1. allowing slots to contain fuzzy sets as values, in addition to text, list, and numeric values,
2. allowing partial inheritance through is-a slots.

As a consequence of the representation of imprecise and uncertain knowledge, it is necessary to develop adequate reasoning methods. Since 1973, when Zadeh suggested the compositional rule of inference, a lot of work has been done in the field of fuzzy inference mechanisms [Dubois and Prade 1988a, p. 67; Zimmermann 1988, p. 736].

Nevertheless, there does not yet exist—and probably never will—a generally usable expert system shell that can be applied to all possible contexts. One of the reasons is that human reasoning depends on the context, i.e., the person with a specific educational background and the situation in which a problem has to be solved. The selection of existing models for the "implication" in chapter 9 is one indication of this. There are essentially two ways to circumvent this difficulty: Either a fuzzy expert system shell has to be designed for a small subset of contexts (i.e., medical diagnosis problems, technical diagnosis, or management planning problems) or such a shell will be a toolbox including various ways of reasoning, uncertainty representations, linguistic approximation, etc., from which the appropriate approaches have to be selected in a certain context. Since the second version does not yet exist, we shall turn towards considering exemplarily some more dedicated expert systems.

10.3 Applications

We shall now illustrate the use of fuzzy set theory in expert systems by sketching some example "cases" (existing expert systems and published approaches that could be used in systems).

Case 10–1: Linguistic Description of Human Judgments [Freksa 1982]

Freksa presents empirical results that suggest that more natural, especially linguistic representations of cognitive observations yield more informative and reliable interpretations than do traditional arithmomorphic representations. He starts from the following assumed chain of cognitive transformations.

$$\begin{array}{c}\text{object}\\ \downarrow\\ \text{percept}\\ \downarrow\\ \text{mental representation}\\ \downarrow\\ \text{verbal description}\\ \downarrow\\ \text{formal description}\\ \downarrow\\ \text{interpretation.}\end{array}$$

The suggested representation system for "soft observations" is supposed to have the following properties [Freksa 1982, p. 302]:

1. The resolution of the representation should be flexible to account for varying precision of individual observations.
2. The boundaries of the representing objects should not necessarily be sharp and should be allowed to overlap with other representing objects.
3. Comparison between different levels of resolution of representation should be possible.
4. Comparison between subjective observations of different observers should be possible.
5. The representation should have a small "cognitive distance" to the observation.
6. It should be possible to construct representing objects empirically rather than from theoretical considerations.

The observations are expressed by simple fuzzy sets that can be described by the quadruples $\{A, B, C, D\}$, illustrated in figure 10–3, with the following interpretation: It is entirely possible that the actual feature value observed is in the range $[B, C]$; it may be possible that the actual value is in the ranges $[A, B]$ or $[C, D]$, but more easily closer to $[B, C]$ than further away; an actual value outside of $[A, D]$ is incompatible with the observation. $[B, C]$ is called "core," and $[A, B]$ and $[C, D]$ are called "penumbra" of the possibility distribution.

The construction of a repertoire of semantic representations for linguistic descriptors is done in the following way (see figure 10–4):

1. The observer selects a set of linguistic labels that allows for referencing all possible values of the feature dimension to be described.

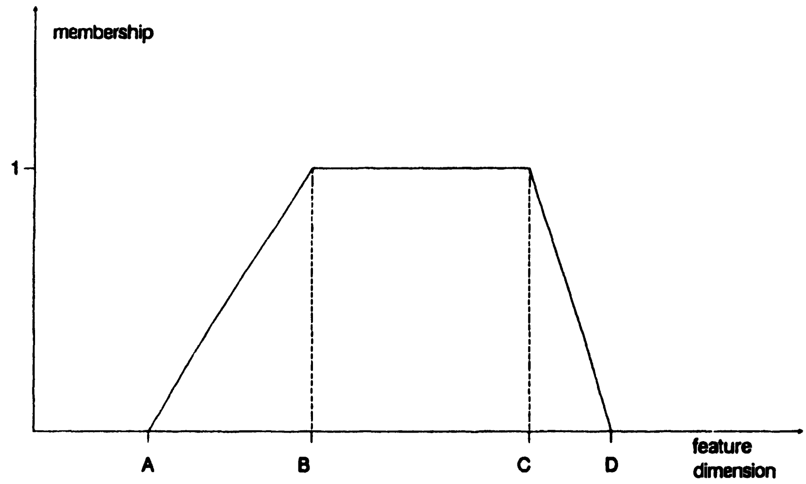


Figure 10–3. Linguistic descriptors.

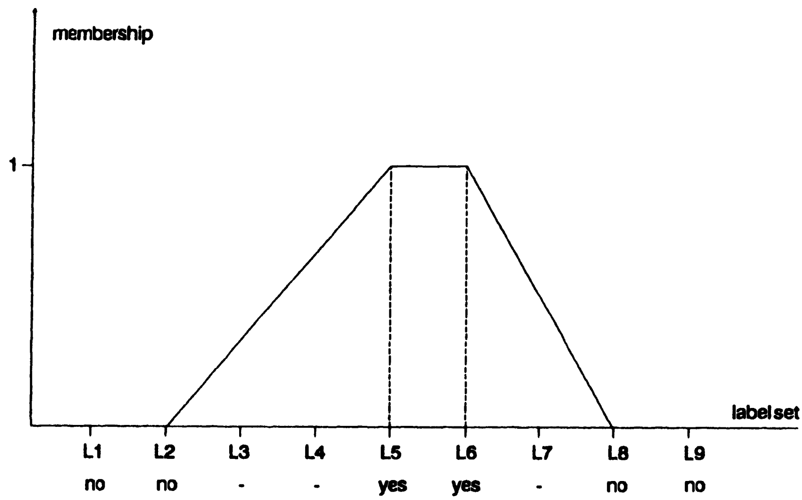


Figure 10–4. Label sets for semantic representation.

2. The repertoire of linguistic labels is arranged linearily or hierarchically in accordance with their relative meaning in the given feature dimensions.
3. A set of examples containing a representative variety of feature values in the given feature dimension is presented to the observer. The observer marks all linguistic labels that definitively apply to the example feature value with "yes" and the labels that definitely do not apply with "no." The labels that have not been marked may be applicable, but to a lesser extent than the ones marked "yes."
4. From the data thus obtained, simple membership functions are constructed by arranging the example objects according to their feature values (using the same criterion by which the linguistic labels had been arranged). These values form the domain for the assignment for membership values.
5. Finally, we assign to a given label the membership value "yes" to the range of examples in which the given label was marked "yes" for all examples and the membership value "no" to the ranges in which the given label was marked "no" for all examples. The break-off points between the regions with membership value "no" and "yes" are connected by some continuous, strongly monotonic function to indicate that the membership of label assignment increases the closer one gets to the region with membership assignment "yes" [Freska 1982, p. 303].

It is not difficult to imagine how the above technique could be used in expert systems for knowledge acquisition and for the user interface.

Case 10–2: CADIAG-2, An Expert System for Medical Diagnosis

Expert knowledge in medicine is to a large extent vagus. The use of objective measurements for diagnostic purposes is only possible to a certain degree. The assignment of laboratory test results to the ranges "normal" or "pathological" is arbitrary in borderline cases, and many observations are very subjective. The intensity of pain, for instance, can only be described verbally and depends very much on the subjective estimation of the patient. Even the relationship between symptoms and diseases is generally far from crisp and unique. Adlassnig and Kolarz [1982, p. 220] mention a few typical statements from medical books that should illustrate to readers who are not medical doctors the character of available information:

Acute pancreatitis is almost always connected with sickness and vomiting.

Typically, acute pancreatitis begins with sudden aches in the abdomen.

The case history frequently reports about ulcus ventriculi and duodendi.

Bilirubinurie excludes the hemolytic icterus but bilirubin is detectable with hepatocellular or cholestatic icterus.

They designed and implemented CADIAG-2, for which they stated the following objectives [Adlassnig 1980, p. 143; Adlassnig et al. 1985]:

1. Medical knowledge should be stored as logical relationships between symptoms and diagnoses.
2. The logical relationship might be fuzzy. They are not obliged to correspond to Boolean logic.
3. Frequent as well as rare diseases are offered after analyzing the patient's symptom pattern.
4. The diagnostic process can be performed iteratively.
5. Both proposals for further investigations of the patient and reasons for all diagnostic results are put out on request.

To sketch their system, let us use the following symbols:

$$\tilde{S} = \{\tilde{S}_1, \ldots, \tilde{S}_m\} := \text{set of symptoms}$$
$$\tilde{D} = \{\tilde{D}_1, \ldots, \tilde{D}_n\} := \text{set of diseases or diagnoses}$$
$$\tilde{P} = \{\tilde{P}_1, \ldots, \tilde{P}_q\} := \text{set of patients}$$

All $\tilde{S}_i$, $\tilde{D}_j$, and $\tilde{P}_k$ are fuzzy sets characterized by their respective membership functions.

$\mu_{\tilde{S}_i}$ expressed the intensity of symptom i

$\mu_{\tilde{D}_j}$ expresses the degree of membership of a patient to $\tilde{D}_j$

$\mu_{\tilde{P}_k}$ assigns to each diagnosis a degree of membership for $\tilde{P}_k$.

Two aspects of symptom $\tilde{S}_i$ with respect to disease $\tilde{D}_j$ are of particular interest:

1. *Occurrence* of $\tilde{S}_i$ in case of $\tilde{D}_j$, and
2. *Confirmability* of $\tilde{S}_i$ for $\tilde{D}_j$

This leads to the definition of two fuzzy sets:

$$\tilde{O}(x), \quad x = \{0, 1, \ldots, 100\} \quad \text{for occurrence of } \tilde{S}_i \text{ at } \tilde{D}_j$$

and

$$\tilde{C}(x), \quad x = \{0, 1, \ldots, 100\} \quad \text{representing the frequency with which } \tilde{S}_i \text{ has been confirmed for } \tilde{D}_j$$

The membership functions for these two fuzzy sets are defined to be

$$\mu_{\tilde{O}}(x) = f(x; 1, 50, 99) \quad x \in X$$
$$\mu_{\tilde{C}}(x) = f(x; 1, 50, 99) \quad x \in Y$$

where X is the occurrence space, Y is the confirmability space, and f is defined as follows (see also figure 9–4!):

$$f(x; a, b, c) = \begin{cases} 0 & x \leq a \\ 2\left(\dfrac{x-a}{c-a}\right)^2 & a < x \leq b \\ 1-2\left(\dfrac{x-c}{c-a}\right)^2 & \text{for } b < x \leq c \\ 1 & \text{for } x > c \end{cases}$$

The $\tilde{S}_i\tilde{D}_j$ occurrence and confirmability relationships are acquired empirically from medical experts using the following linguistic variables:

i	*Occurrence* $\tilde{O}_i$	*Confirmability* $\tilde{C}_i$
1	always	always
2	almost always	almost always
3	very often	very often
4	often	often
5	unspecific	unspecific
6	seldom	seldom
7	very seldom	very seldom
8	almost never	almost never
9	never	never
	unknown	unknown

The membership functions of $\tilde{O}_i$ and $\tilde{C}_i$ are shown in figure 10–5. They are arrived at by applying modifiers (see definition 9–3) to "never" and "always." For details of the data acquisition process, see Adlassnig and Kolarz [1982, p. 226].

Other relationships such as symptom–symptom, disease–disease, and symptom–disease are also defined as fuzzy sets (fuzzy relations). Possibilistic interpretations of relations (min-max) are used. Given a patient's symptom pattern, the symptom | disease relationships, the symptom | combination-disease relationships, and the disease | disease relationships yield fuzzy diagnostic

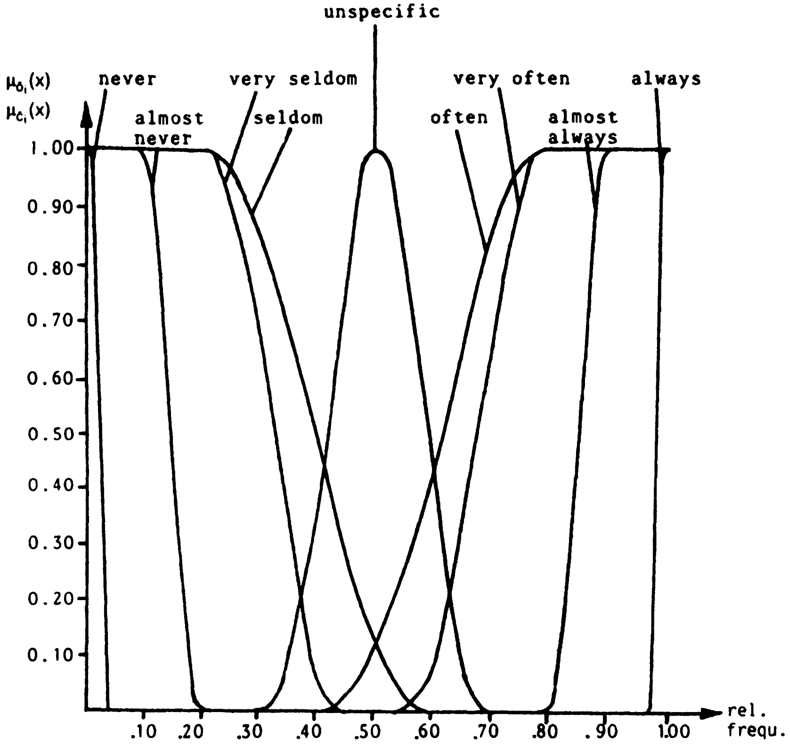


Figure 10–5. Linguistic variables for occurrence and confirmability.

indications that are the basis for establishing confirmed and excluded diagnosis as well as diagnostic hypotheses.

Three binary fuzzy relations are then introduced: The occurrence relation, $\tilde{R}_{\tilde{O}}$, the confirmability relation, $\tilde{R}_{\tilde{C}}$, *both in* $X \otimes Y$, and the symptom relation, $\tilde{R}_{\tilde{S}_i}$, which is determined on the basis of the symptom patterns of the patients.

Finally, four different fuzzy indications are calculated by means of fuzzy relation compositions [Adlassnig and Kolarz 1982, p. 237]:

1. $\tilde{S}_i\tilde{D}_j$ *occurrence indication* $\tilde{R}_1 = \tilde{R}_{\tilde{S}} \circ \tilde{R}_{\tilde{O}}$

$$\mu_{\tilde{R}_1}(p, \tilde{D}_j) = \max_{\tilde{S}_i} \min\{\mu_{\tilde{R}_{\tilde{S}}}(p, \tilde{S}_j), \mu_{\tilde{R}_{\tilde{O}}}(\tilde{S}_i, \tilde{D}_j)\}$$

2. $\tilde{S}_i\tilde{D}_j$ *confirmability indication* $\tilde{R}_2 = \tilde{R}_{\tilde{S}} \circ \tilde{R}_{\tilde{C}}$

$$\mu_{\tilde{R}_2}(p, \tilde{D}_j) = \max_{\tilde{S}_i} \min\{\mu_{\tilde{R}_{\tilde{S}}}(p, \tilde{S}_i), \mu_{\tilde{R}_{\tilde{C}}}(\tilde{S}_i, \tilde{D}_j)\}$$

3. $\tilde{S}_i\tilde{D}_j$ *nonoccurrence indication* $\tilde{R}_3 = \tilde{R}_{\tilde{S}} \circ (1 - \tilde{R}_{\tilde{O}})$

$$\mu_{\tilde{R}_3}(p, \tilde{D}_j) = \max_{\tilde{S}_i} \min\{\mu_{\tilde{R}\tilde{S}}(p, \tilde{S}_i), 1 - \mu_{\tilde{R}\tilde{O}}(\tilde{S}_i, \tilde{D}_j)\}$$

4. $\tilde{S}_i\tilde{D}_j$ *nonsymptom indication* $\tilde{R}_4 = (1 - \tilde{R}_{\tilde{S}}) \circ \tilde{R}_{\tilde{O}}$

$$\mu_{\tilde{R}_4}(p, \tilde{D}_j) = \max_{\tilde{S}_i} \min\{1 - \mu_{\tilde{R}\tilde{S}}(p, \tilde{S}_i), \mu_{\tilde{R}\tilde{O}}(\tilde{S}_i, \tilde{D}_j)\}$$

Similar indications are determined for symptom |disease relationships, and we arrive at 12 fuzzy relationships $\tilde{R}_j$.

Three categories of diagnostic relationships are distinguished:

1. Confirmed diagnoses
2. Excluded diagnoses
3. Diagnostic hypotheses

Diagnoses are considered confirmed if

$$\mu_{\tilde{R}_j} = 1 \quad \text{for} \quad j = 1 \text{ or } 6$$

or if the max-min composition of them yields 1.

For excluded diagnosis, the decision rules are more involved; and for diagnostic hypotheses, all diagnoses are used for which the maximum of the following pairs of degrees of membership are smaller than .5:

$$\max\{\mu_{\tilde{R}_j}, \mu_{\tilde{R}_k}\} \leq .5 \quad \text{for}$$
$$\{j, k\} = \{1, 2\} \quad \text{or} \quad \{5, 6\} \quad \text{or} \quad \{9, 10\}$$

CADIAG-2 can be used for different purposes: for example, diagnosing diseases, obtaining hints for further examinations of patients, and explanation of patient symptoms by diagnostic results.

Case 10–3: SPERIL I, an Expert System to Assess Structural Damage
[Ishizuka et al. 1982]

Earthquake engineering has become an important discipline in areas in which the risk of earthquake is quite high.

> Frequently, the safety and reliability of a particular or a number of existing structures need to be evaluated either as part of a periodic inspection program or immediately following a given hazardous event. Because only a few experienced engineers can practice it well to date, it is planned to establish a systematic way for the damage assessment of existing structures. SPERIL is a computerized damage assessment system as

designed by the authors particularly for building structures subjected to earthquake excitation [Ishizuka et al. 1982, p. 262]

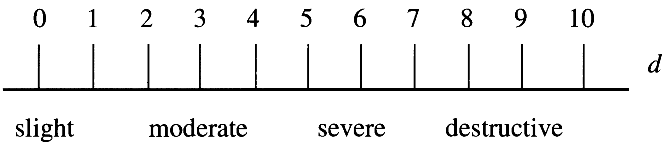
Useful information for the damage assessment comes mainly from the following two sources:

1. visual inspection at various portions of the structure
2. analysis of accelerometer records during the earthquake

The interpretation of these data is influenced to a large extent by the particular kind of structure under study. Information for damage assessment is usually collected in a framework depicted in figure 10–6.

It is practically impossible to express the inferential knowledge of damage assessment precisely. Therefore the production rules in SPERIL I are fuzzy. A two-stage procedure is used to arrive at fuzzy sets representing the degree of damage. First the damage is assessed on a 10-point scale, and then the rating is transformed into a set of terms of the linguistic variable "damage."

Let d be the damage evaluated at a 10-point scale. Then the relationship between the terms and the original ratings can be described as follows:



$$\tilde{T}_{\text{no}}(d) = \{(0,1),(1,.5)\}$$
$$\tilde{T}_{\text{slight}}(d) = \{(1,.5),(2,.1),(3,.5)\}$$
$$\tilde{T}_{\text{moderate}}(d) = \{(3,.5),(4,.1),(5,.7),(6,.3)\}$$
$$\tilde{T}_{\text{severe}}(d) = \{(5,.3),(6,.7),(7,1),(8,.7),(9,.3)\}$$
$$\tilde{T}_{\text{destructive}}(d) = \{(8,.3),(9,.7),(10,1)\}$$

The rule associated with node 2 in figure 10–8, for instance, would then read

IF:	MAT is reinforced concrete,
THEN IF:	STI is no,
THEN:	GLO is no with 0.6,
ELSE IF:	STI is slight,
THEN:	GLO is slight with 0.6,
ELSE IF:	STI is moderate,
THEN:	GLO is moderate with 0.6,

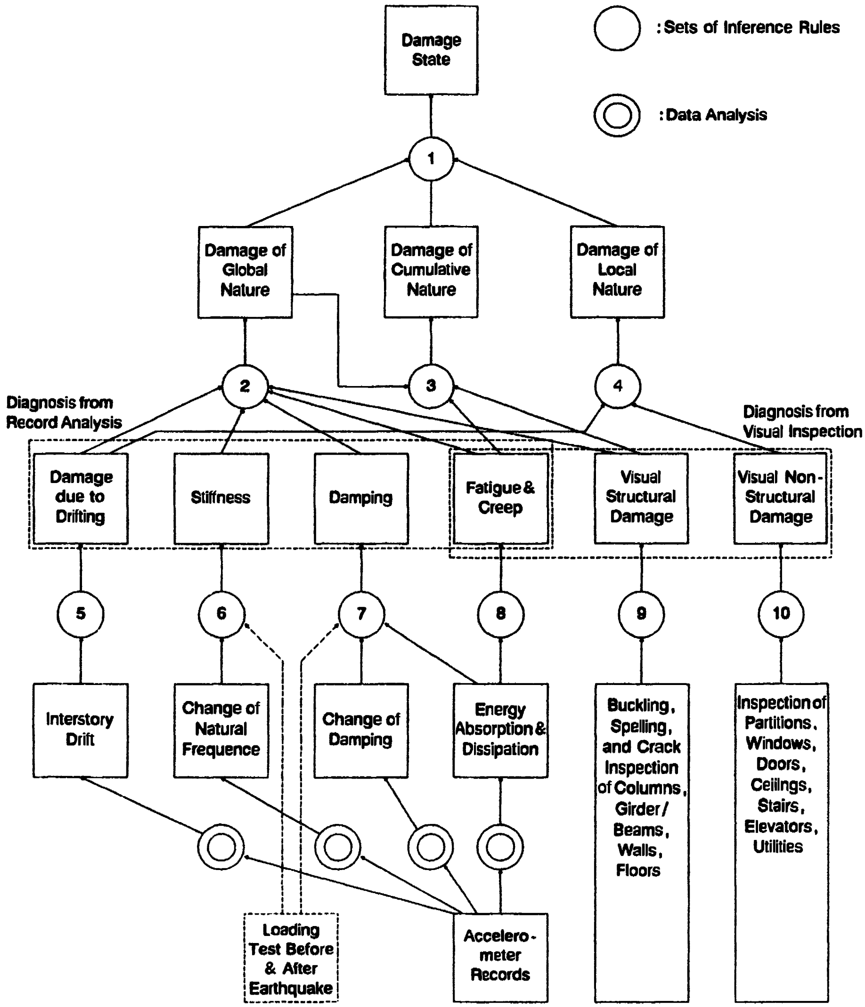


Figure 10–6. Inference network for damage assessment of existing structures [Ishizuka et al. 1982, p. 263].

ELSE IF:	STI is severe,
THEN:	GLO is severe with 0.6,
ELSE IF:	STI is destructive,
THEN:	GLO is destructive with 0.6,
ELSE:	GLO unknown with 1,

where

MAT = structural material,
GLO = damage of global nature,
STI = diagnosis of stiffness, and
"unknown" stands for the universe set of damage grade.

To obtain a correct answer by using such knowledge, a rational inference mechanism is required to process the rules expressed with fuzzy subsets along with uncertainty in an effective manner.

To include uncertainty, first Dempster's and Shafer's probabilities were used [Dempster 1967; Shafer 1976]. Thus the conclusions were accompanied by a lower and upper probability indicating lower and upper bounds of subjective probabilities. (For details, see Ishizuka et al. [1982, pp. 264–266].)

It was felt that the rules as shown for node 2 could not necessarily be expressed as crisp rules. Therefore fuzzy inference rules were introduced in order to arrive at a fuzzy damage assessment together with upper and lower probabilities. For details, the reader is again refereed to the above-mentioned source.

Improvements, particularly of the knowledge acquisition phase, have been suggested [Fu et al. 1982; Watada et al. 1984]. They either use fuzzy clustering or a kind of linguistic approximation.

Case 10–4: ESP, an Expert System for Strategic Planning
[Zimmermann 1989]

Strategic planning is a large heterogeneous area with changing content over time and without a closed theory such as is available in other areas of management and economics. It deals with the long-range planning of a special company and is frequently done for independent autonomous units, called strategic business units (SBUs) [Hax and Majluf 1984, p. 15]. One technique for analyzing the current and future business position is the business portfolio approach.

The original idea of portfolio analysis in strategic planning was to describe the character of a corporation by the positions of SBUs in a two-dimensional portfolio matrix and to try to find strategies aimed at keeping this "portfolio" balanced. Some of the major problems encountered are given below.

Dimensionality. It is obvious that two dimensions are insufficient to describe adequately the strategic position of an SBU. Two dimensions are certainly preferable for didactical reasons and for presentation, but for realistic description a multidimensional vectorial positioning would be better.

Data Collection and Aggregation. Even for a two-dimensional matrix, the dimensions of an SBU must be determined by a rather complex data-gathering and aggregation process. Factors such as ROI, market share, and market growth can be obtained without too much difficult. Other factors to be considered are combinations of many aspects. It is, therefore, not surprising that intuitive aggregation and the use of scoring methods are rather common in this context, although their weaknesses are quite obvious: Aggregation procedures are kept simple for computational efficiency, but they are very often not justifiable. Different factors are considered to be independent without adequate verification. A lot of subjective evaluations enter the analysis with very litte control.

Strategy Assignment. In classical portfolio matrixes, broad strategic categories have been defined to which basic strategies are assigned. It is obvious that these categories are much too rough to really define operational strategies for them. One of the most important factors in determining real strategies will be the knowledge and experience of the strategic planners who transform those very general strategic recommendations into operational strategies—a knowledge that is not captured in the portfolio matrixes!

Modeling and Consideration of Uncertainty. In an area into which many ill-structured factors, weak signals, and subjective evaluations enter, and which extends so far into the future, uncertainty is obviously particularly relevant. Unfortunately, however, uncertainty is hardly considered in most of the strategic planning systems we know. The utmost that is done is to sometimes attach uncertainty factors to an estimate and then to aggregate those together with the data in a rather heuristic and arbitrary way.

ESP, an Expert System for Strategic Planning, tries to improve classical approaches and to remedy some of their shortcomings. It also provides a framework in which strategic planners can analyze strategic information and develop more sophisticated strategic recommendations. It characteristics are as follows:

Dimensionality. Multidimensional portfolio matrixes are used. For visualization, two dimensions each can be chosen; the location of SBUs are defined by vectors, however. As an example, let us consider the four following dimensions:

1. Technology Attactiveness
2. Technology Position
3. Market Attractiveness
4. Competitive Position

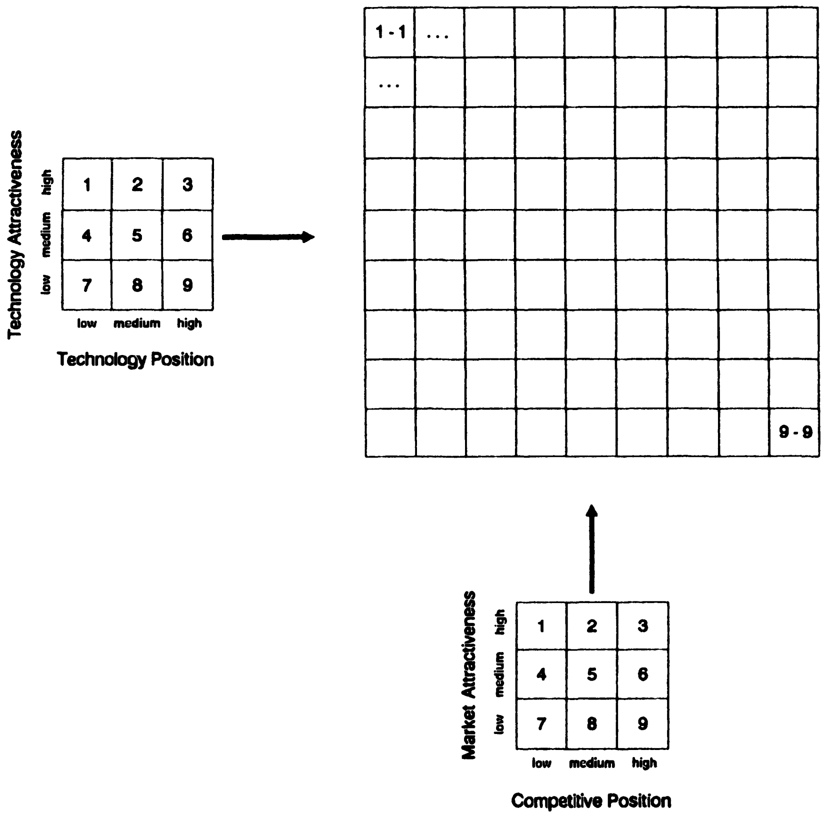


Figure 10–7. Combination of two two-dimensional portfolios.

If we combine the first two and the last two dimensions we obtain two two-dimensional portfolio matrixes which, combined, correspond to a four-dimensional matrix (see figure 10–7). If each of the two-dimensional matrixes consists of nine strategic categories by having three intervals—low, medium, high—on each axis, then the combined matrix contains $9 \times 9 = 81$ strategic positions. Graphically, only the two-dimensional matrixes are shown. The positions of the combined matrix are only stored vectorially and used for more sophisticated policy assignment.

Data Collection and Aggregation. Each "dimension" is defined by a tree of subcriteria and categories. Figure 10–8 shows a part of the tree for "Technology Attractiveness."

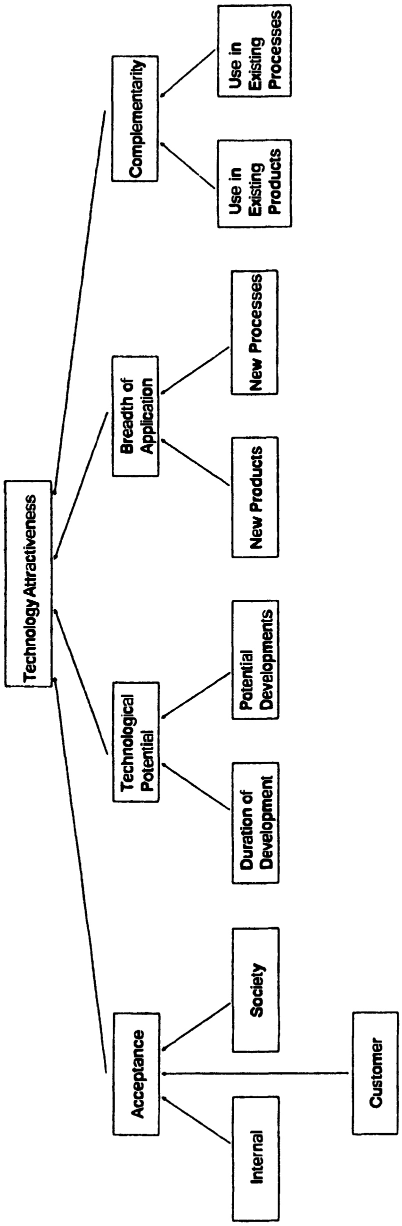


Figure 10–8. Criteria tree for technology attractiveness.

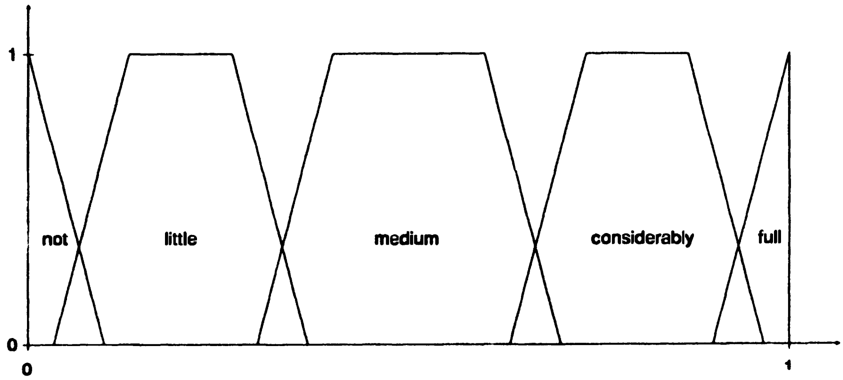


Figure 10–9. Terms of "degree of achievement."

The input given by the user consists of one linguistic variable for all criteria of the leaves (lowest subcriteria in each of the four trees). This linguistic variable denotes the respective "degree of achievement"; it can be chosen from the terms "not at all," "little," "medium," "considerably," and "full." These terms are represented by trapezoidal membership functions that are characterized by their four characteristic values on their supports (see figure 10–9).

To arrive at the root of each tree, these ratings of the leaves are aggregated on every level of the tree by using the γ-operator, described in chapter 3. There the reader will find other operators (e.g. minimum, product), which can also be chosen by the user. It is suggested that this aggregation of linguistic terms, rather than of numerical values, be done by aggregating the four characteristic values of each trapezoid in order to obtain the respective characteristic value of the resulting trapezoid. The last aggregation level of one tree is shown in figure 10–10. Repeating this procedure for all characteristic values of the membership functions of all aggregation steps of each of the four trees leads to a trapezoidal membership function for each of the criteria.

Strategy Assignment. As already mentioned, strategy assignment is made on the basis of the vectorially described position of an SBU. Two levels can be distinguished:

1. General Policy Recommendation
   This is assigned to the position of the SBU as it is defined by the values of the roots of the trees. In our example, the position would be defined by technology attractiveness, technology position, competitive position, and market attractiveness.

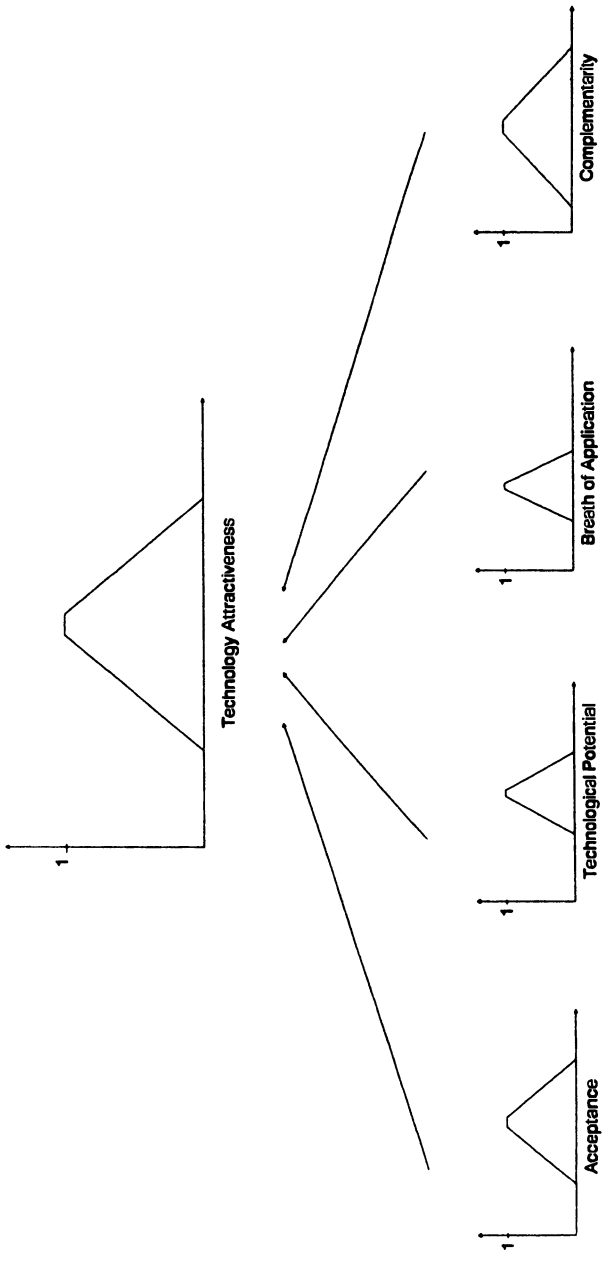


Figure 10–10. Aggregation of linguistic variables.

2. Detailed Policy Recommendation
   Policy recommendations based on the location in the portfolio matrix, which in turn is determined by the values of the roots of the evaluation trees only, can only be very rough guidelines. The same value at a root of the tree can be obtained from very different vectors of values of the nodes of the first level of the tree. The values of this vector are, therefore, used to make more specific strategic recommendations in addition to the basic policy proposal mentioned above. In the example tree shown in figure 10–8, for instance, the ratings of "Acceptance," "Technological Potential," "Breadth of Application," and "Complementarity" would be used for such a specification of the strategic recommendation.

Modeling and Consideration of Uncertainty. It is possible for the user of ESP to interact with this system by defining a special α-level that results in a rectangle in the portfolio matrixes, as shown in figure 10–11. The α-level denotes the desired degree of certainty, and the corresponding area in the matrix is a visualization of the possible position of the considered SBU.

ESP: Implementation. We had intended to design ESP by using one of the available shells. It turned out, however, that none of the available shells offers all the features we needed. Therefore, a combination of a shell (in this case Leonardo 3.15) with a program (in Turbo Pascal) had to be used. The basic structure of ESP is shown in figure 10–12.

Knowledge Base I contains primarily rules that assign basic strategy recommendations to locations of SBU in multidimensional portfolio matrixes and detailed supplementary recommendations to profiles of the first levels of trees. Together with the inference engine, it provides for the user the "if-then" part and the explanatory function. For this part, the shell Leonardo 3.15 was used.

Knowledge Base II contains the structures of the free defineable trees that determine the location of an SBU in the different dimensions of the multidimensional matrix. The "Aggregator" computes their values and characteristic values for the linguistic values for all nodes of the trees on the basis of available structural knowledge (tree structure, α-values, and γ-values) and on the basis of data (μ-values) entered for each terminal leaf by the user. The information provided by the "Aggregator" is then used for the visual presentation of two-dimensional matrixes and profiles and also supports the explanatory module.

All aggregation and visual presentation functions could not be accommodated by Leonardo 3.15. Therefore, an extra program in Turbo-Pascal and the appropriate bridge programs to Leonardo had to be written.

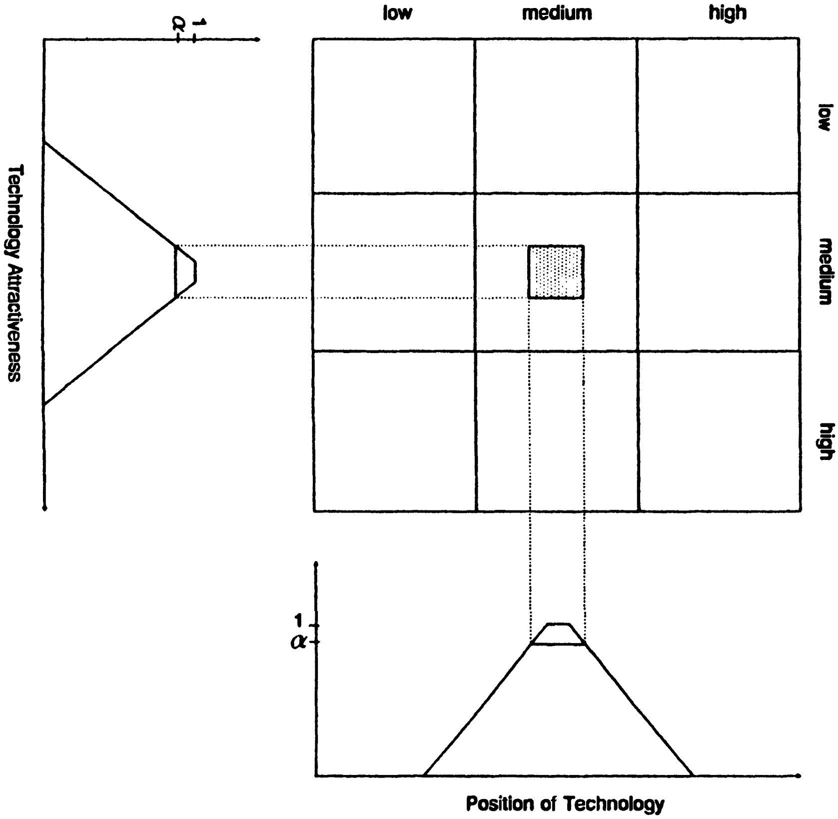


Figure 10–11. Portfolio with linguistic input.

ESP is fully menu driven. It could be considered as a second-generation expert system that works with shallow knowledge (KB I) as well as with deep knowledge (KB II).

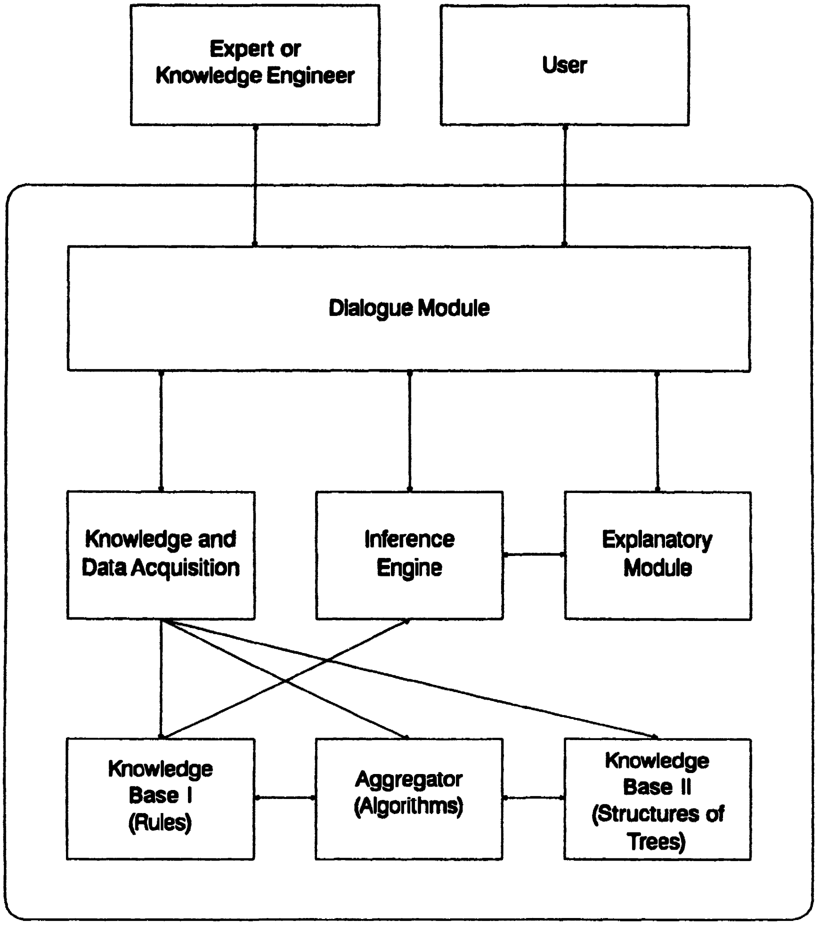


Figure 10–12. Structure of ESP.

Exercises

1. What are the differences between a decision support system and an expert system?
2. Construct examples of domain knowledge represented in the form of rules, frames, and networks. Discuss advantages and disadvantages of these three approaches.
3. List, describe, and define at least four different types of uncertainty mentioned in this book. Associate appropriate theoretical approaches with them.

4. An expert in strategic planning has evaluated linguistically the degree of achievement of the lowest subcriteria of the criterion "Technology Attractiveness." He denotes the corresponding trapezoidal membership functions by the vectors of the characteristic values. After the first aggregation step, the evaluation of the first-level criteria results. The respective trapezoidal membership functions are given by the following vectors of the characteristic values:

   Acceptance: (.2, .3, .5, .7)

   Technological Potential: (.6, .7, .9, 1)

   Breadth of Application: (.4, .5, .6, .7)

   Complementarity: (.1, .3, .4, .6)

   Compute the four characteristic values of the criterion "Technology Attractiveness" by using the γ operator with $\gamma = .5$ and equal weights for all first-level criteria for the four respective characteristic values given above. Draw the resulting stripe in the portfolio matrix for $\alpha = .8$.

11 FUZZY CONTROL

11.1 Origin and Objective

The objective of fuzzy logic control (FLC) systems is to control complex processes by means of human experience. Thus fuzzy control systems and expert systems both stem from the same origins. However, their important differences should not be neglected. Whereas expert systems try to exploit uncertain knowledge acquired from an expert to support users in a certain domain, FLC systems as we consider them here are designed for the control of technical processes. The complexity of these processes range from cameras [Wakami and Terai 1993] and vacuum cleaners [Wakami and Terai 1993] to cement kilns [Larsen 1981], model cars [Sugeno and Nishida 1985], and trains [Yasunobu and Miamoto 1985]. Furthermore, fuzzy control methods have shifted from the original translation of human experience into control rules to a more engineering-oriented approach, where the goal is to tune the controller until the behavior is sufficient, regardless of any human-like behavior.

Conventional (nonfuzzy) control systems are designed with the help of physical models of the considered process. The design of appropriate models is time-consuming and requires a solid theoretical background of the engineer. Since modeling is a process of abstraction, the model is always a simplified version of the process. Errors are dealt with by means of noise signals, supplementary model states, etc. Many processes can, however, be controlled by humans without any model, and there are processes that cannot be controlled with conventional control systems but are accessible to control by human operators—for example, most people with a driving licence can drive a car without any model. The formalization of the operator's experience by the methods of fuzzy logic was the main idea behind fuzzy logic control:

> The basic idea behind this approach was to incorporate the "experience" of a human process operator in the design of the controller. From a set of linguistic rules which describe the operator's control strategy a control algorithm is constructed where the words are defined as fuzzy sets. The main advantages of this approach seem to be the possibility of implementing "rule of the thumb" experience, intuition, heuristics, and the fact that it does not need a model of the process [Kickert and Mamdani 1978, p. 29].

Almost all designers of FLC systems agree that the theoretical origin of those systems is the paper "Outline of a New Approach to the Analysis of Complex Systems and Decision Processes" by Zadeh [1973b]. It plays almost the same role that the Bellman–Zadeh [1970] paper titled "Decision Making in a Fuzzy Environment" does for the area of decision analysis. In particular, the compositional rule of inference (see definition 9–7) is considered to be the spine of all FLC models. The original activities centered around Queen Mary College in London. Key to that development was the work of E. Mamdani and his students in the Department of Electrical and Electronic Engineering. Richard Tong, of nearby Cambridge, was another key figure in the development of fuzzy control theory. The first application of fuzzy set theory to the control of systems was by Mamdani and Assilian [1975], who reported on the control of a laboratory model steam engine. It is interesting to note that the first industrial application of fuzzy control was the control of a cement kiln in Denmark [Holmblad and Ostergaard 1982]. The area of fuzzy control was neglected by most European and American control engineers and managers until the end of the 1980s, when Japanese manufacturers launched a wide range of products with fuzzy controlled parts and systems.

Fuzzy control was (and still is) treated with mistrust by many control engineers. This attitude towards fuzzy control is changing, and most of the progress in this area is due to control engineers who started with conventional control theory (and still apply it). "Fuzzy logic" became a marketing argument in Japan at the end of the 1980s, and popular press articles gave the impression that fuzzy control systems are cheap, easy to design, very robust, and capable of outperforming conventional control systems. This is certainly not generally true; the real situation depends heavily on the system to be controlled. The lack of prac-

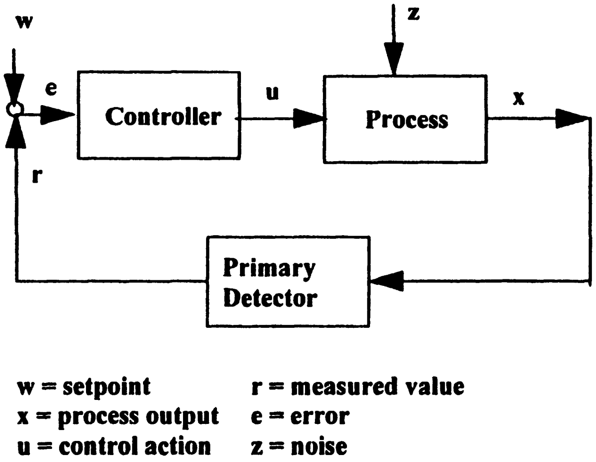


Figure 11–1. Automatic feedback control.

tical experience in FLC design and well-trained engineers in the field must also be considered when one decides to implement fuzzy controllers. FLC is, however, beginning to establish itself as a recognized control paradigm and will play a major role in control theory in the future.

11.2 Automatic Control

The process of automatic control of a technical process relies mainly on the comparison of desired states of the process with some measured or evaluated states. The controller tries to reach the desired states (setpoints) by adjustment of the input values of the process that are identical to the translated output values of the controller. Due to the continuous comparison of these values, one gets a closed-loop system. Usually a noise signal leads to deviations from the set-points and thus to dynamically changing controller outputs. Figure 11–1 depicts an automatic feedback control system.

Conventional control strategies use process models or experimental results as a basis for the design of the control strategies. The well-known PID controllers are widely used design paradigms. They use information about the input–output behavior of the process to generate the control action. The behavior of the closed loop is controlled by different gain values that can be adjusted independently by the control engineer. Modern computer-controlled (direct digital control, DDC) systems have to deal with sampled values and are therefore modeled as time-discrete control systems with sampling units. Thus the control action is a function of the error vector of recent errors e: = $[e(k), e(k-1), \ldots, e(k-r)]$, where k is the sampling time, and the vector of the last control outputs u: = $[u(k-1), u(k-2), \ldots, u(k-s)]$. We derive the current control action as $u(k)$: $= f(e, u)$. Note that $e(k)$ and $u(k)$ can be vectors in systems with many inputs and outputs (MIMO).

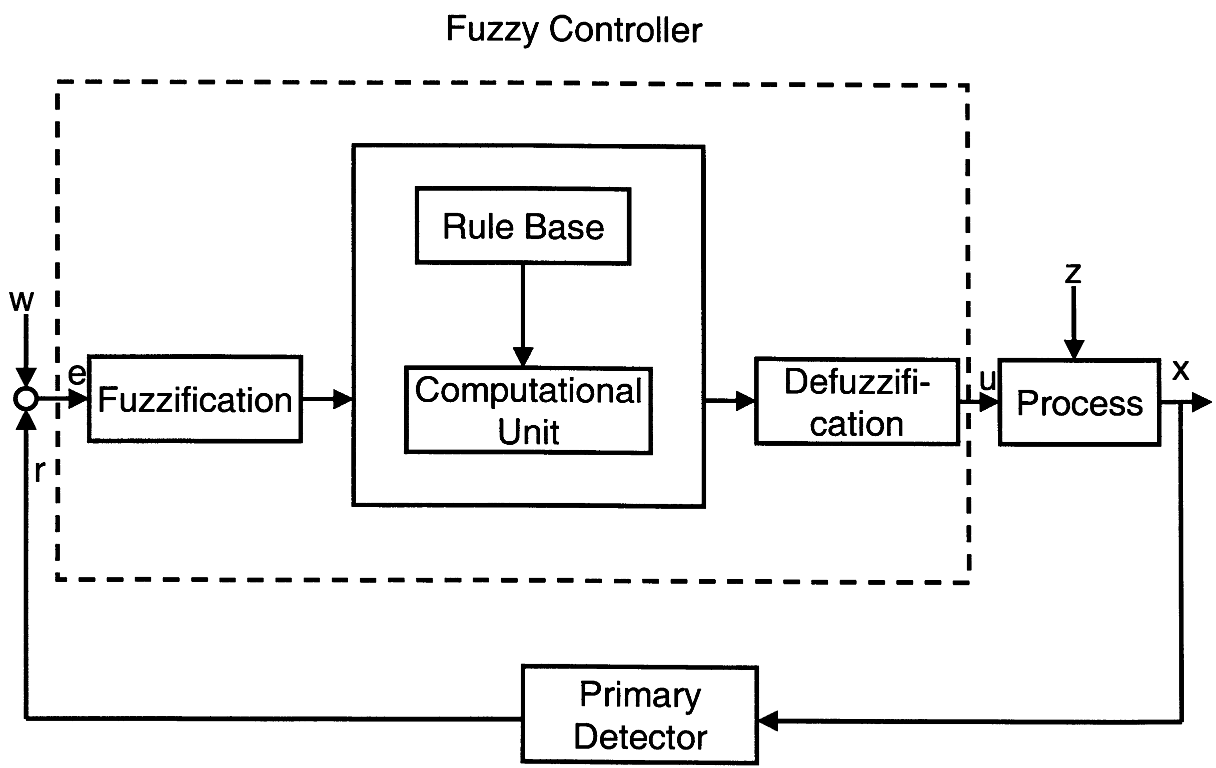
11.3 The Fuzzy Controller

Fuzzy controllers are special DDC systems that use rules to model process knowledge in an explicit way. Instead of designing algorithms that explicitly define the control action as a function of the controller input variables, the designer of a fuzzy controller writes rules that link the input variables with the control variables by terms of linguistic variables. Consider, for example, the heating system in your living room. If the temperature is *slightly too low*, then you would probably want to increase the heating power *a bit*. If you now want to control the room temperature by a fuzzy controller, you just interpret the terms "slightly too low" and "a bit" as terms of linguistic variables and write *rules* that link these variables, e.g.,

If temp = "slightly too low,"
then change of power = "increased by a bit"

After all rules have been defined, the control process starts with the computation of all rule-consequences. Then the consequences are aggregated into one fuzzy set describing the possible control actions, which in this case are different values of the change of power. These computations are done with the *computational unit*. Since our heating system doesn't understand a control action like "increased by a bit," the corresponding fuzzy set has to be defuzzified into one crisp control action using the *defuzzification* module. This simple example illustrates the main ingrediences of a fuzzy controller: the rule base that operates on linguistic variables, the fuzzification module that generates terms as functions of the crisp input values (temperature, in this case), and the computational unit that generates the terms of the output variables as a function of the input terms and the rules of the rule base. Since the controlled process has to be fed with a crisp signal (instead of increased by a bit in the example), the result of the computational unit that is a term of a linguistic variable has to be transformed into a crisp value. Figure 11–2 depicts a generic so-called "Mamdani" fuzzy controller. Modifications of this scheme are possible and will be explicitly discussed later.

When designing fuzzy controllers, several decisions regarding the structure and the methodology have to be made. It is possible to view a fuzzy controller



w = setpoint **r = measured value**
x = process output **e = error**
u = control action **z = noise**

Figure 11–2. Generic "Mamdani" fuzzy controller.

as a 7-tuple with the entries (input/fuzzification/rules/rule evaluation/aggregation/defuzzification/output) [compare Buckley 1992]. Possible decision parameters are as follows:

Input: number of input signals, number of derived states of each input signal, scaling of the input signal

Fuzzification: type of membership functions, mean, spread and peak of membership functions, symmetry, crosspoints, continuous or discrete support, granularity (number of membership functions)

Rules: number of rules, number of antecedents, structure of rule base, type of membership functions in consequences, rule weights

Rule evaluation: aggregation operator in the antecedent, inference operator

Aggregation: aggregation operator combining the results of the individual rules, individual rule-based inference (functional approach), or composition-based inference (relational approach)

Defuzzification: defuzzification procedure

Output: number of output signals (usually determined by problem structure), scaling

We will return to these parameters in more detail later. This classification, however, shows that a fuzzy controller is the result of a sequence of decisions by the designer. It is therefore not appropriate to talk about *the* fuzzy controller; one should rather explicate which type of controller is under consideration. Many modifications of Mamdani's original controller [Mamdani and Assilian 1975] have been proposed since the publication of the original paper in 1975. One important and often used modification was introduced by Sugeno [1985b] and will be described after the discussion of Mamdani's original controller.

11.4 Types of Fuzzy Controllers

11.4.1 The Mamdani Controller

The main idea of the Mamdani controller is to describe process states by means of linguistic variables and to use these variables as inputs to control rules. We start with the assignment of terms to input variables. The base variable is an input variable that can be measured or derived from a measured signal or an output variable of the controller. In the heating system example, possible base variables

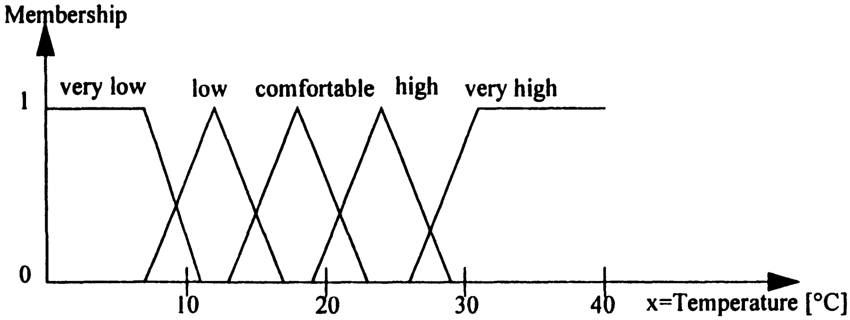


Figure 11–3. Linguistic variable "Temperature."

are room temperature, change of room temperature, number of open windows, outdoor temperature, change of power, etc. This example illustrates that the number of input signals is far from obvious. The terms of the linguistic variables are fuzzy sets with a certain shape. It is popular to use trapezoidal or triangular fuzzy sets due to computational efficiency, but other shapes are possible. The linguistic variable "temperature" could, for example, consist of the terms "very low" (vl), "low" (l), "comfortable" (c), "high" (h), and "very high" (vh), as shown in figure 11–3.

Formally, we describe the terms of each linguistic variable $LV_1, \ldots, LV_n$ by their membership functions $\mu_i^j(x)$, where i indicates the linguistic variable, $i = 1, \ldots, n$; j indicates the term of the linguistic variable i, $j = 1, \ldots, m(i)$, and $m(i)$ is the number of terms of the linguistic variable i. The number of linguistic variables and the number of terms of each linguistic variable determine the number of possible rules. In most applications, certain states can be neglected either because they are impossible or because a control action would not be helpful. It is therefore sufficient to write rules that cover only parts of the state space.

The rules connect the input variables with the output variables and are based on the fuzzy state description that is obtained by the definition of the linguistic variables. Formally, the rules can be written as

$$\text{rule } r\text{: if } x_1 \text{ is } A_1^{j_1} \text{ and } x_2 \text{ is } A_2^{j_2} \text{ and} \ldots \text{and } x_n \text{ is } A_n^{j_n}\text{, then } u \text{ is } A^j$$

where $A_i^{j_i}$ is the jth term of linguistic variable i corresponding to the membership function $\mu_i^{j_i}(x_i)$ and A^j corresponds to the membership function $\mu^j(u)$ representing a term of the control action variable. A reasonable rule in the heating system example is

if temperature is low and change_of_temperature is negative small,
then power is medium

Table 11–1. Rule base.

temp/change_te	*nb*	*ns*	*z*	*ps*	*pb*
vl		b	b	m	m
l	b	m	m	s	s
c		m	s	s	
h		s	s	s	
vh	m	s	s		

The rule base in systems with two inputs and one output can be visualized by a rule table where the rows and columns are partitioned according to the terms of the input variables and the entries are the rule consequences. Assume that we have defined five terms of the linguistic variable "change_of_temperature": "negative big" (nb), "negative small" (ns), "zero" (z), "positive small" (ps), "positive big" (pb), and three control action terms for the "power": "small" (s), "medium" (m) and "big" (b). A possible rule base is then visualized in table 11–1. Empty entries refer to states with no explicitly defined rules. The first empty entry (vl, nb) in table 11–1 refers to a state where the temperature is very low and falling rapidly. Since the heating system has limited power, even maximal power would not lead to a comfortable temperature. A rule that covers this situation is therefore superfluous. One should, however, define a default value that is used as a controller output if neither of the rules fires.

The definition of linguistic variables and rules are the main design steps when implementing a Mamdani controller. Before elaborating on the last design step, which is the choice of an appropriate defuzzification procedure, we show how input values trigger the computation of the control action. The computational core can be described as a three-step process consisting of

1. determination of the degree of membership of the input in the rule-antecedent,
2. computation of the rule consequences, and
3. aggregation of rule consequences to the fuzzy set "control action."

The first step is to compute the degrees of membership of the input values in the rule antecedents. Employing the minimum-operator as a model for the "and," we compute the degree of match of rule r as

$$\alpha_r = \min_{i=1,\ldots,n}\{\mu_i^{j_i}(x_i^{\text{input}})\}$$

This concept enables us to obtain the validity of the rule consequences. We assume that rules with a low degree of membership in the antecedent also have little validity and therefore clip the consequence fuzzy sets at the height of the antecedent degree of membership. Formally,

$$\mu_r^{\text{conseq}}(u) = \min\{\alpha_r, \mu^j(u)\}$$

The result of this evaluation process is obtained by aggregation of all consequences using the maximum operator. We compute the fuzzy set of the control action:

$$\mu^{\text{conseq}}(u) = \max_r\{\ \mu_r^{\text{conseq}}(u)\}$$

This computation is a special case of an inference process described in chapter 10, and other inference methods can be applied. It is important to note that Mamdani's method takes into account all rules in a single stage and that no chaining occurs. Thus the inference process in fuzzy control is much simpler than in most expert systems.

In our heating system example, we assume that the current temperature is 22°C and that the change_of_temperature is –0.6°C/min. Thus we get that temperature is "comfortable" with degree 0.4 and "high" with degree 0.3 (see figure 11–3). A similar definition of the linguistic variables in the change_of_temperature case yields "negative small" with degree 0.6 and "zero" with degree 0.2. In table 11–1, we see that four rules have a degree of match greater than zero:

r10: if temp = "comfortable" and change_of temp = "negative small," then power = "medium"

r11: if temp = "comfortable" and change_of temp = "zero," then power = "small"

r13: if temp = "high" and change_of temp = "negative small," then power = "small"

r14: if temp = "high" and change_of temp = "zero," then power = "small"

The degree of membership is

$$\begin{aligned}\alpha_{10} &= \min\{0.4, 0.6\} = 0.4\\ \alpha_{11} &= \min\{0.4, 0.2\} = 0.2\\ \alpha_{13} &= \min\{0.3, 0.6\} = 0.3\\ \alpha_{14} &= \min\{0.3, 0.2\} = 0.2\end{aligned}$$

Accordingly, the consequences of the rules are

$$\begin{aligned}\mu_{10}^{\text{conseq}}(u) &= \min\{0.4, \mu^{\text{medium}}(u)\}\\ \mu_{11}^{\text{conseq}}(u) &= \min\{0.2, \mu^{\text{small}}(u)\}\\ \mu_{13}^{\text{conseq}}(u) &= \min\{0.3, \mu^{\text{small}}(u)\}\\ \mu_{14}^{\text{conseq}}(u) &= \min\{0.2, \mu^{\text{small}}(u)\}\end{aligned}$$

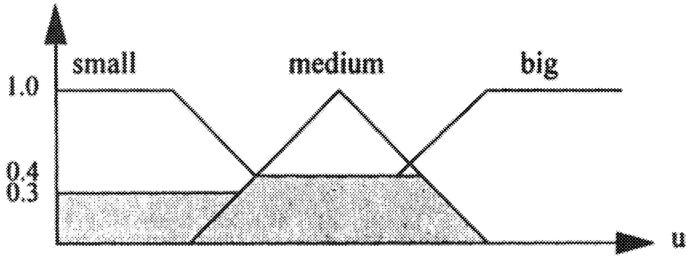


Figure 11–4. Rule consequences in the heating system example.

Figure 11–4 depicts the resulting fuzzy set of control action

$$\mu^{\text{conseq}}(u) = \max\{\mu_{10}^{\text{conseq}}(u), \mu_{11}^{\text{conseq}}(u), \mu_{13}^{\text{conseq}}(u), \mu_{14}^{\text{conseq}}(u)\}$$

11.4.2 Defuzzification

Since technical processes require crisp control actions, a procedure that generates a crisp value out of one or more given fuzzy (output) sets is required. These defuzzification methods are very often based on heuristic ideas, such as, "take the action that corresponds to the maximum membership", "take the action that is midway between two peaks or at the center of the plateau", etc. Of course, these methods can also be characterized by their formal (mathematical) properties. Also, defuzzification is not only relevant for fuzzy control but also for other types of problems, e.g. multi criteria analysis (see chapter 14) and other areas in which fuzzy sets have to be transformed into crisp expressions (real numbers, symbols, etc.). We discuss it here in the context of fuzzy control because historically it became first relevant in this context.

In this book we will describe and discuss the best known defuzzification strategies and analyze their main properties and interrelationships. For many other defuzzification approaches that exist, the reader is referred to references where they are discussed in detail. (See, for instance, [Lee 1990; Runkler and Glesner 1993, 1994; Driankov 1993; Yager and Filev 1994; Yager 1996; Runkler 1996; Li 1996; van Leekwijck and Kerre 1999]).

The crisp value to be chosen should generally be an element of the supports of the fuzzy sets to be defuzzified. The criteria, however, which are used to find this element can depend on very different bases: it can be the type of inference of which the fuzzy set is a result of (see [Li 1996], it can be special points of the membership functions (e.g. maxima or minima), it can be the area below the membership functions or it can be other indicators.

In decision making, for instance, we want to achieve semantical correctness. This means to define "characteristic" or "significant" elements which are probably those that have highest membership (maxima) [Runkler and Glesner 1994]. In fuzzy control we are looking for the most important rule base entry which might require to take into consideration weights of the rules etc.

Other criteria for the choice of the defuzzification method is the scale level on which the membership function is available (see chapter 16).

In the following we will first describe some elementary and some extended defuzzification methods and then compare them with respect to their properties.

Extreme Value Strategies. These defuzzification strategies use extremal values of the membership function (generally the maxima) to define the crisp equivalent value. Let us assume that the membership function is not unimodal (have a unique maximum) but either have several maxima with the same value of $\mu(x)$ or a "core", i.e. a compact subset of the support in which the degree of membership has the maximum value (a plateau as maximum). Depending on whether the left, the right end or the center of the "core" is considered most appropriate for defuzzification, one arrives at one of the following strategies:

Left of maximum (LOM)
Right of maximum (ROM), or
Center of maximum (COM).

Definition 11–1

The *core* of a fuzzy set is defined as

$$\mathrm{Co}(x) = \{x | x \in X \quad \text{and} \quad \neg(\exists y \in X)(A(y) > A(x))\}$$

Then for the LOM-strategy the defuzzified value is

$$u_{LOM} = \min\{u | u \in Co\}$$

For the ROM-strategy it is

$$u_{ROM} = \max\{u | u \in Co\}$$

and for the COM-strategy it is

$$u_{COM} = \frac{u_{ROM} - u_{LOM}}{2}$$

This should not be confused with the "Mean of Maxima" (MOM) strategy, which assumes that there is not a core of the fuzzy sets but separate different maxima.

Figure 11–5 depicts the above three strategies for our example.

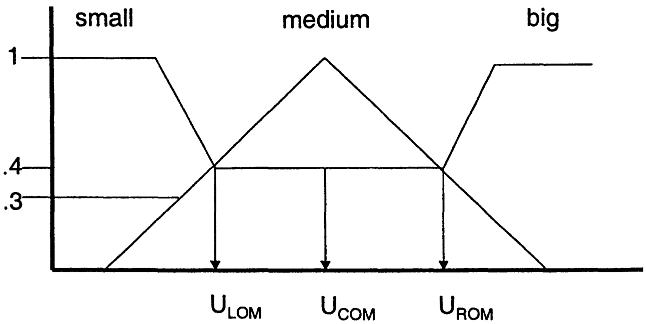


Figure 11–5. Extreme value strategies.

Controid Strategies (Area Methods). The information taken into account in above strategies is very limited. If more information shall be considered, which is available via the membership function of the fuzzy set to be defuzzified, then one normally resorts to centroid strategies. The best-known of these are the "center of areas" and the "center of gravity" strategies.

Center of Area. The COA method chooses the control action that corresponds to the center of the area with membership greater than zero. The idea of this method is to aggregate the information about possible control actions that is represented by the membership function. The solution is a compromise, due to the fuzziness of the consequences. Formally, the control action is computed as: The defuzzified value is the support element that divides the area below a continuous membership function into two equal parts.

$$\int_{x_{\min}}^{d_{COA}} \mu(x)dx = \int_{d_{COA}}^{x_{\max}} \mu(x)dx$$

The procedure can be computationally complex and can lead to unwanted results if the fuzzy set is not unimodal. The result of the COA defuzzification for the heating system example is depicted in figure 11–6.

The *center of gravity* (COG) method is the most trivial weighted average and has a distinct geometrical meaning, that is the center of gravity or center of mass. From a mathematical point of view the COG corresponds to the expected value of probability. It is defined as

$$u_{COG} = \frac{\int_u u \cdot \mu(u)du}{\int_u \mu(u)du}$$

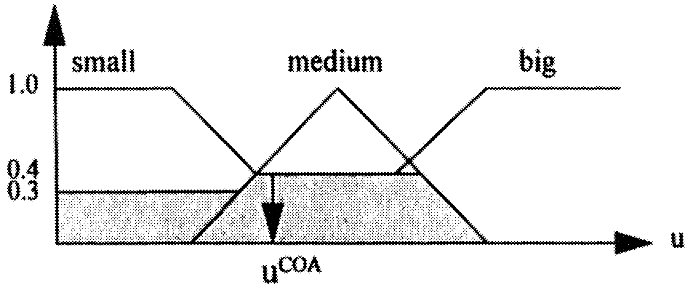


Figure 11–6. COA defuzzification.

All those defuzzification strategies might lead to problems if the fuzzy set to be fuzzified is not compact, i.e. if it really consists of several fuzzy sets in between of them there are "forbidden" zones. These are intervals of the action space which do not belong to the support and from which no element should be chosen as a defuzzified action. This can, for instance, happen if a car approaches an obstacle and two possible (fuzzy) strategies are: "turn slightly right" and "turn slightly life". The defuzzified strategy would most likely be "go straight ahead", which is obviously not very desirable.

Example 11–1 [Runkler and Glesner 1993]

Let us assume a heating system which can be run at high or low degrees (but not in-between). The total range (universe) is u = [0,255], and the two relevant rules of the inference engine have weights of h and (1 – h).

We shall consider two situations for changing weights: neighboring and separate membership functions.

Since there is no unique maximum LOM, ROM and COM would only consider the "core" and would, therefore, always stay in "low" for h > (1 – h) and in "high" for h < (1 – h). For h = .5 the defuzzified values would be extremely different for LOM and ROM. They would in any case not change continuously with h.

For COA and COG they would, for h = .5, even be at 127, certainly a not very desirable value.

Let us now consider the situation shown in figure 11–8.

The range of $63 \leq u \leq 127$ is the "forbidden zone".

For LOM, ROM we would at least stay off the forbidden zone, but for COM (and for h = .5) we would certainly end up in it. For COA and COG we would also find defuzzified values in the forbidden zone for large ranges of h.

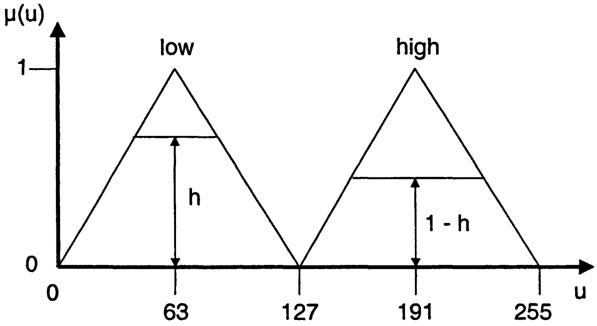


Figure 11–7. Neighboring membership functions.

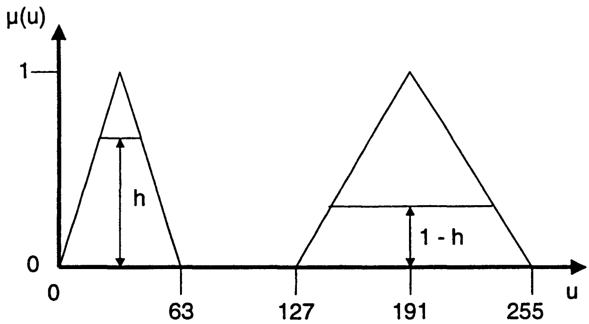


Figure 11–8. Separate membership functions.

These undesirable effects can be avoided by using parameterized defuzzification strategies, such as "Extended Center of Area" (XCOA) or "Extended Center of Gravity" (XCOG).

Exemplarily we will show the XCOA strategy [Runkler and Glesner 1993]:

$$u^{XCOA} = \frac{\int_{S1} (\mu(u))^{\alpha(u)}\, du}{\int_{S2} (\mu(u))^{\alpha(u)}\, du},$$

where S1 and S2, respectively, are the supports of the two fuzzy sets.

This strategy reduces to

Mean Value of supports	for	$\alpha = 0$
COA	for	$\alpha = 1$
MOM	for	$\alpha \to \infty$

For the situation of example 11–1 XCOA jumps for low values of α from 191 to 63. For high values of α it behaves as MOM. For medium α the defuzzified value slowly slides to the edge of the forbidden zone and then jumps over it to the opposite edge of the forbidden zone. It never lies in it!

Scale levels and properties of defuzzifiers

Obviously for nominal scale levels of the universe (see type A membership model in chapter 16) a defuzzification does not make sense at all. The first scale level from which a defuzzification makes sense at all is an ordinal scale level of the universe. Generally a cardinal scale level (interval, ratio or absolute scale) would have to be required.

For the membership functions there are similar requirements. The views, however, on which scale levels membership functions are supplied in practice diverge considerably (see also chapter 16).

For the defuzzifying strategies some authors [Runkler 1996; Li 1996; van Leekwijck and Kerre 1999] have also various desirable properties.

From all these suggested we will select in the following the most important ones and those with respect to which the defuzzification strategies we have discussed differ at all:

Property 1: *Closed Property*
The defuzzified value of a fuzzy set should be an element of its support.

Property 2: *Fuzzy Singleton*
If a fuzzy set has a positive degree of membership for only one element, then the defuzzification should select this element.

Property 3: *Horizontal Movement*
If a fuzzy set is shifted horizontally by a distance d, the defuzzified value should make the same movement.

Property 4: *(Strong) Monotony*
Monotony in this context means, that, if D(A) is the defuzzified value of the fuzzy set A, and the degrees of membership are increased on one side of D(A), then D(A) should move to this side.

Property 5: *Balance*
If a fuzzy set is enlarged or reduced on both sides of D(A), then D(A) should not change.

Table 11–2. Properties of defuzzifiers.

	Property									
Strategy	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
LOM	Y	Y	Y	Y	No	Y	No	No	Yes	No
ROM	Y	Y	Y	Y	No	Y	No	No	Yes	No
COM	No	Y	Y	Y	No	Y	Y	No	Yes	No
COA	No	Y	Y	Y	Y	No	Y	No	No	Y
COG	No	Y	Y	Y	No	No	Y	No	No	Y
XCOA	Y	Y	Y	Y	Y	No	Y	No	No	Y

Property 6: *Strong Vertical Translation*
The defuzzified value stays unchanged if a constant is added to all membership values.

Property 7: *Equality*
If two convex fuzzy sets A and B have the same level center curves, then they should have the same defuzzified value. Here "level center curves" are curves that divide each α-level of a fuzzy set in two equal parts.

Property 8: *T-norm property*
If two fuzzy sets A and B are combined by a t-norm, then the defuzzified value of $A \in B$ should be in the interval bounded by the defuzzified values of the two fuzzy sets.

Property 9: *T-conorm property*
If two fuzzy sets A and B are combined by a t-conorm, the defuzzified value of this combined fuzzy set should be in the interval bounded by the defuzzified values of A and B.

Property 10: *Continuity*
A small variation in any of the degrees of membership should not result in a big change of the defuzzified value.

Table 11–2 shows which of the described defuzzification strategies has which property.

So far we have discussed formal mathematical properties which can be valuable when deciding which defuzzification should be used. In addition, however, other criteria may turn out to be important.

1. **Computational Effort.** Is the method slow or fast when implemented as an algorithm? Does the situation require a fast method (for instance, in on-

line embedded control) or is time not a relevant dimension (e.g. often in decision making)?

2. **Inference.** Do we want the defuzzification to take into consideration the type of inference we are using or shall it even adapt to changes in the inference engine?

3. **Plausibility.** Does the defuzzification method yield a plausible control action and is it stable or oversensitive?

Other criteria are possible (see, e.g., Driankov et al. [1993] and Pfluger, Yen, and Langari [1992]) and depend on the application under consideration. The choice of an appropriate defuzzification procedure can therefore be compared to the choice of an appropriate aggregation operator as discussed in chapter 3.

11.4.3 The Sugeno Controller

An often-used modification of Mamdani's controller was presented by Sugeno [1985b] and Sugeno and Nishida [1985]. The idea is to write rules that have fuzzy antecedents, equivalent to the Mamdani controller, and crisp consequences that are functions of the input variables. The rule results are aggregated as weighted sums of the control actions corresponding to each rule. The weight of each rule is the degree of membership of the input value in the rule antecedent as computed in the Mamdani controller. A defuzzification procedure is therefore superfluous. A rule can formally be written as

rule r: if x_1 is $A_1^{j_1}$ and x_2 is $A_2^{j_2}$ and . . . and x_n is $A_n^{j_n}$, then u is $f_r(x_1, x_2, \ldots, x_n)$

where the variables are defined as in the Mamdani case. The consequence function, which depends on the input variables, is usually linear, but other types may be used. In the heating system example, we may write a rule like

if temperature is low and change_of_temperature is negative small
then power = 400 − 120 · temp-23 · delta_temp [W]

The definition of a functional relationship is not straightforward but allows the identification of parameter values in the consequence function.

The control action is computed with the help of the degrees of membership that are evaluated exactly as in the Mamdani controller. We obtain

$$u^{\text{Sugeno}} = \frac{\sum_r \alpha_r \cdot f_r(x_1, x_2, \ldots, x_n)}{\sum_r \alpha_r}$$

It is possible to view the linear Sugeno controller as a linear controller that is valid around a fuzzily defined operating point. The control algorithm in the operating point is perfectly valid and loses validity with decreasing degree of membership, which is computed with the help of the rule antecedents. Thus the control strategy is a combination of several linear control strategies defined at different points in the state space.

11.5 Design Parameters

The design of a fuzzy controller involves decisions about a number of important design parameters that can be determined before the actual control starts and/or on-line. Important design parameters are the fuzzy sets in the rules, the rules themselves, scaling factors in input and output, inference methods, and defuzzification procedures. Although other design parameters also play important roles, we want to focus on the parameters that have to be defined in almost all control applications. Defuzzification has already been discussed thoroughly and inference is discussed in connection with expert systems (chapter 10).

11.5.1 Scaling Factors

The easiest-to-change parameters are the scaling factors. The scaling factors scale the base variables of the linguistic variables. Formally, the input and output variables are calculated as $x_i' = sf_i \cdot x_i$, where the x_i' is the variable that is used in the rule and sf_i is the scaling factor of rule i. Scaling factors allow the definition of normalized base variables of the corresponding linguistic variables and play a role similar to the gain in conventional control systems. It is obvious that alternation of the scaling factors has a significant impact on the closed loop behavior of an FLC system.

11.5.2 Fuzzy Sets

The fuzzy sets describe terms of linguistic variables. When the shape of the fuzzy sets are determined, several other parameters have to be adjusted. Here, we will assume that the membership functions have a triangular shape, which by no means is necessary but is often done in fuzzy control applications. The *modal value* or *peak value* of a membership function is the value of the base variable where the membership function is equal to one. The *left* and *right width* of the membership function is the first value of the base variable on the left or right side

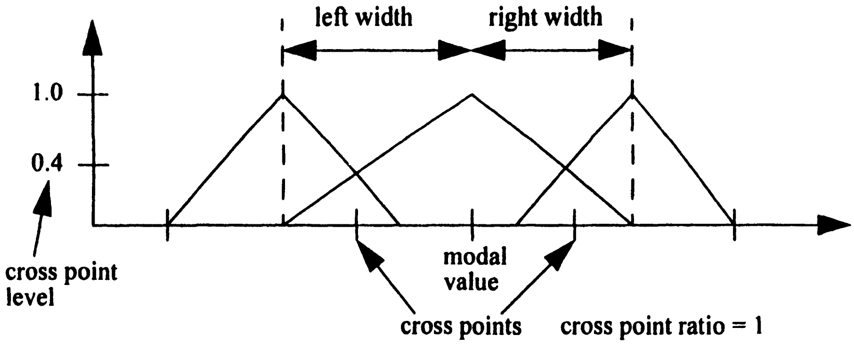


Figure 11–9. Parameters describing the fuzzy sets.

of the peak value, respectively, that has a zero membership. The *cross point* between two membership functions is the value of the base variable where both membership functions assume the same membership value greater than zero. The *cross point level* is the membership at the cross point. Clearly, two membership functions may have more than one cross point. We therefore define the *cross point ratio* as the number of cross points between two membership functions. Figure 11–9 depicts a linguistic variable with three fuzzy sets and the corresponding parameters.

Several rules of thumb can be formulated using the above definitions. The reader should, however, be aware of the empirical character of these rules, i.e., there are no globally valid proofs showing their validity. A common rule claims that all values of the base variable should have a membership greater than zero in at least one membership function corresponding to one of the terms. It is also usual to demand that two adjacent membership functions interact, i.e., that the crosspoint ratio is equal to one for those membership functions. It is therefore often assumed that the *cross point value* between neighboring membership functions is *equal to one* and that the *cross point level is 0.5* [Driankov et al. 1993, p. 120].

Next, we will focus on *symmetry*, which is achieved if the left and the right width are equal. Assume that we have designed a fuzzy controller with a single input, a single rule with a one-term linguistic variable in the consequence, and COA defuzzification. Then the Mamdani controller will clip the membership function of the rule consequence in the height of the membership function in the rule antecedent. If the input matches the rule antecedent with membership one, then we would expect to get the peak value of the rule consequence. This would only be the case if the membership function of the rule consequence is

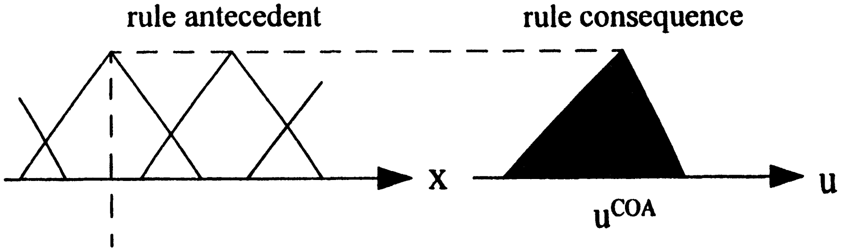


Figure 11–10. Influence of symmetry.

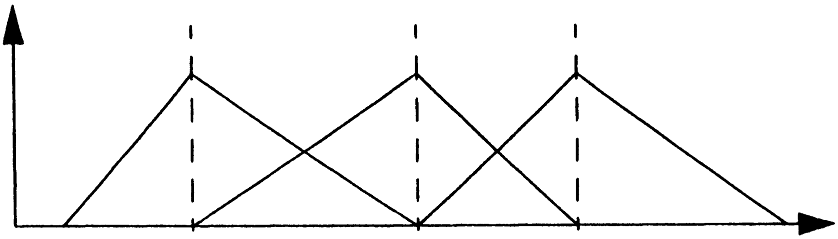


Figure 11–11. Condition width.

symmetrical. This dependency is shown in figure 11–10 for a non symmetrical fuzzy set in the rule consequence.

The *condition width* states that the left-width of the right membership function is equal to the right-width of the left membership function and that they are both equal to the length of the interval between the peak values of the two adjacent membership functions [Driankov et al. 1993, p. 122]. This rule yields smoothly changing control values and avoids large steps. A linguistic variable that satisfies this condition is shown in figure 11–11.

11.5.3 Rules

The entire knowledge of the system designer about the process to be controlled is stored as rules in the knowledge base. Thus the rules have a basic influence on the closed-loop behavior of the system and should therefore be acquired thoroughly. The development of rules may be time-consuming, and designers often have to translate process knowledge into appropriate rules. Sugeno and Nishida [1985] mention four ways to find fuzzy control rules:

1. the operator's experience
2. the control engineer's knowledge
3. fuzzy modeling of the operator's control actions
4. fuzzy modeling of the process

We add the following sources that may also be used:

5. crisp modeling of the process
6. heuristic design rules
7. on-line adaptation of the rules

Usually a combination of some of these methods is necessary to obtain good results. As in conventional control, increased experience in the design of fuzzy controllers leads to decreasing development times.

11.6 Adaptive Fuzzy Control

Many processes have time-variant parameters due to continuous alternation of the process itself. This well-known phenomenon has led to the development of adaptive controllers that change their control behavior as the process changes. This adjustment is called adaptation. It is natural for adaptive fuzzy controllers to change the same controller parameters that a designer may change. Therefore most adaptive FLC systems change the shape of the membership functions, the scaling factors, etc. It is common to distinguish between controllers that modify their rules; these are called *self-organizing* controllers [Procyk and Mamdani 1979], and *self-tuning* controllers [e.g., Bartolini et al. 1982] that modify essentially the fuzzy set definitions. Since adaptive controllers work automatically, a monitor has to be found that detects changes in the process. Two common methods can be distinguished:

1. The *performance measure* approach, where the closed-loop behavior is evaluated by certain performance criteria such as overshoot, rise-time, etc.
2. The *parameter estimator* approach, where a process model is continuously updated due to sampled process information.

It is usually easier to define appropriate performance measures than to find process models that can be updated continuously and that are valid over a wide range of the state space. An overview of the area of adaptive fuzzy controllers is given by Driankov et al. [1993], and researchers continue to work actively in the field. Popular design methods currently include the combination of fuzzy controllers with neural network methods [e.g., Berenji 1992; Berenji and Khedar 1992] and genetic algorithms [e.g., Hopf and Klawonn 1993; Lee and Tagaki 1993].

11.7 Applications

Fuzzy control certainly is the branch of fuzzy set theory with the most applications, and their number is steadily growing. The application boom was started by Japanese manufacturers who applied fuzzy logic to processes ranging from home appliances to industrial control. The first major book containing applications of FLC was edited by M. Sugeno [1985a] and shows that the term "fuzzy control" is not narrowly interpreted as applications of the Mamdani or Sugeno controller to a certain process but includes other fuzzy logic techniques such as fuzzy algebra as well. It is also worthwhile to mention that most successful applications combine FLC systems with conventional control strategies to hybrid systems.

We now present several applications of fuzzy control without going into detail. Interested readers may consult the original literature.

11.7.1 Crane Control

Cranes are widely used in industrial assembly systems where heavy loads have to be transported. Today, modern cranes reach a top speed of 160 m/min and an acceleration of up to $2\,m/s^2$ [Behr 1994]. A container crane is depicted in figure 11–12. One of the main problems that have to be taken into account in the control of such a crane system is that the load may start to swing. This can be avoided with the help of mechanical constructions such as telescopes and stays or electronic loss control. These methods are, however, expensive, and the construction depends on the specific crane under consideration. In contrast, it was observed that an experienced operator was able to control a crane satisfactorily without such advanced devices. This was the motivation for the design of an FLC system for crane control.

The crane control depends on the mode of operation: one distinguishes between *manual operation*, where an operator controls the crane and the objective of the fuzzy controller is to avoid swinging, and *automatic operation*, where a certain position has to be reached. Here we focus on automatic operation.

The automatic operation mode can be divided into three different phases of motion: acceleration, normal motion, and positioning. Figure 11–13 depicts the typical behavior of the speed in the different phases.

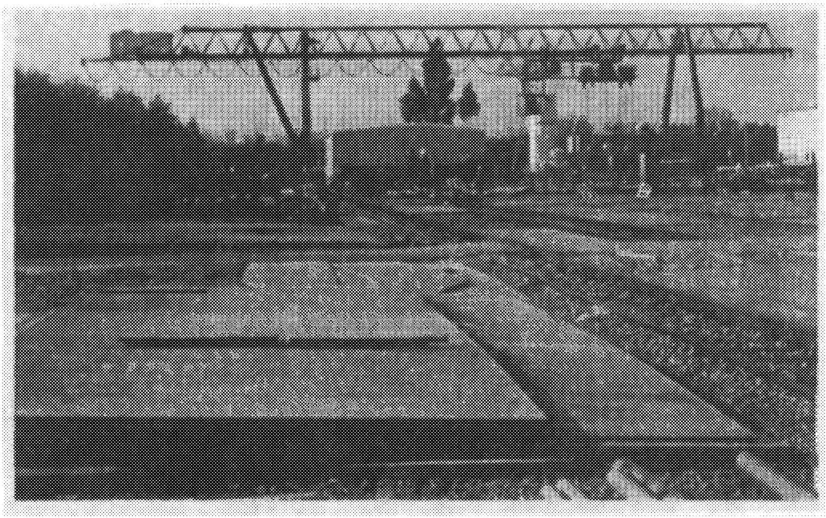


Figure 11–12. Container crane [von Altrock 1993].

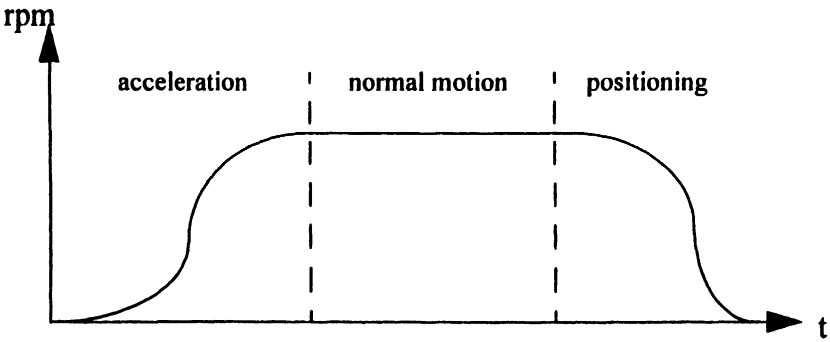


Figure 11–13. Phases of motion.

Different controllers were designed for the three phases. Input values were the position, the speed, the length of the pendulum, the angle of the pendulum, and in some cases the mass of the load. When the mass was unknown, a fuzzy estimator system was activated that calculates the mass as a function of the observed system behavior. The controllers were implemented on a fuzzy processor for real-time control of the crane.

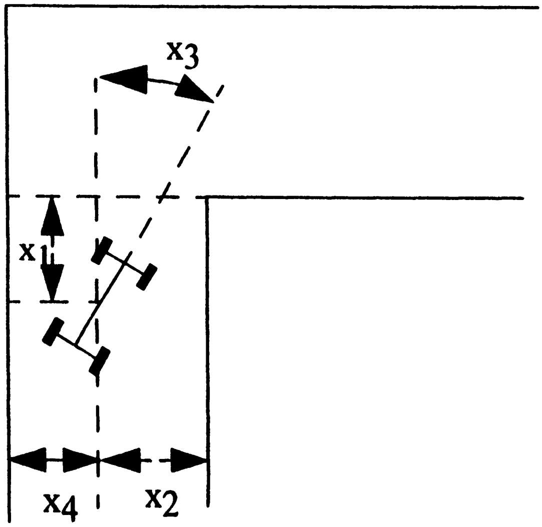


Figure 11–14. Input variables [Sugeno and Nishida 1985, p. 106].

11.7.2 Control of a Model Car

One of the most difficult processes to control with conventional control methods is a car. The mathematical models are large and nonlinear, and simple controllers such as PID controllers do not yield satisfactory results. Most people can, however, drive a car without any mathematical model, and it is clear that they use their knowledge to control the car.

Sugeno and Nishida [1985] were the first to implement and publish the results they obtained with a fuzzy-controlled model car. The fuzzy control rules were derived by modeling an expert's driving actions. Four input variables were used: x_1 = distance from entrance of corner, x_2 = distance from inner wall, x_3 = direction (angle) of car, and x_4 = distance from outer wall. The four variables are depicted in figure 11–14.

These four input variables are used as inputs to a Sugeno controller with 20 rules. The results were very encouraging and are depicted in figure 11–15. It is worthwhile to mention that all rules were derived from an experienced driver's control actions with an identification procedure.

Whereas the study by Sugeno and Nishida treated static problems von Altrock et al. [1992] considered the control of a model car in extreme situations that are inherently dynamic. Typical dynamical situations are sliding and skidding. The model

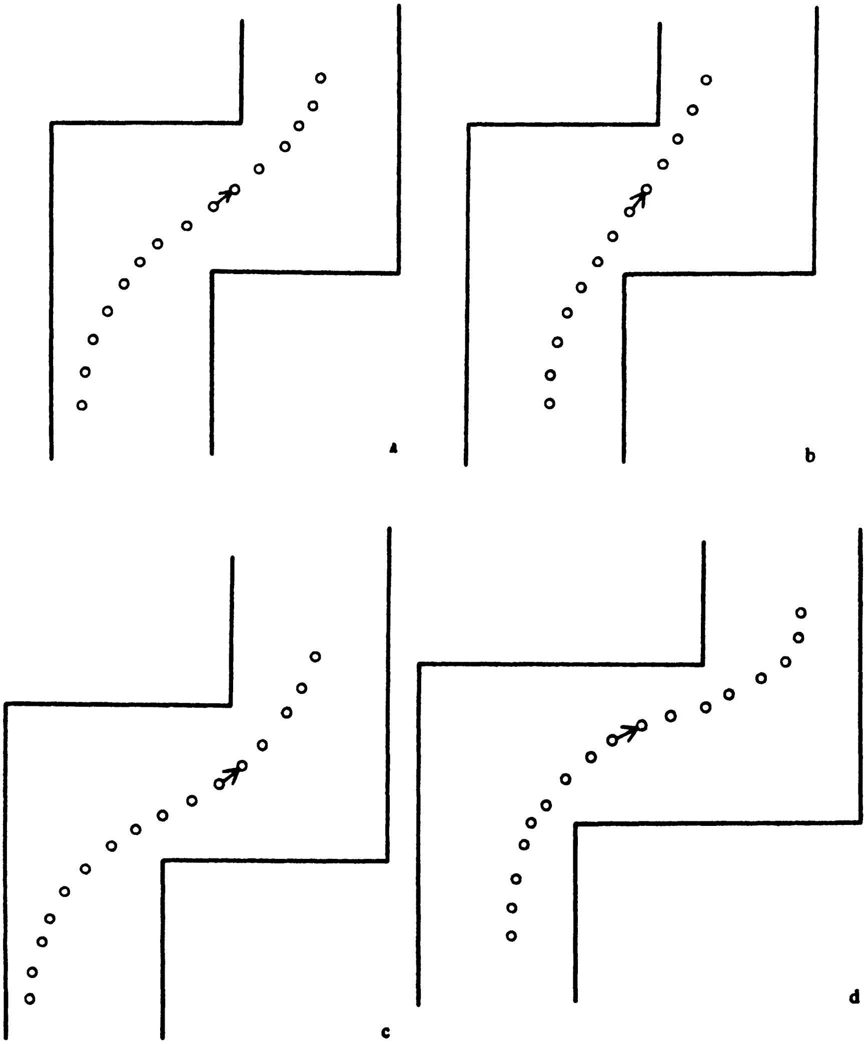


Figure 11–15. Trajectories of the fuzzy controlled model car [Sugeno and Nishida 1985, p. 112].

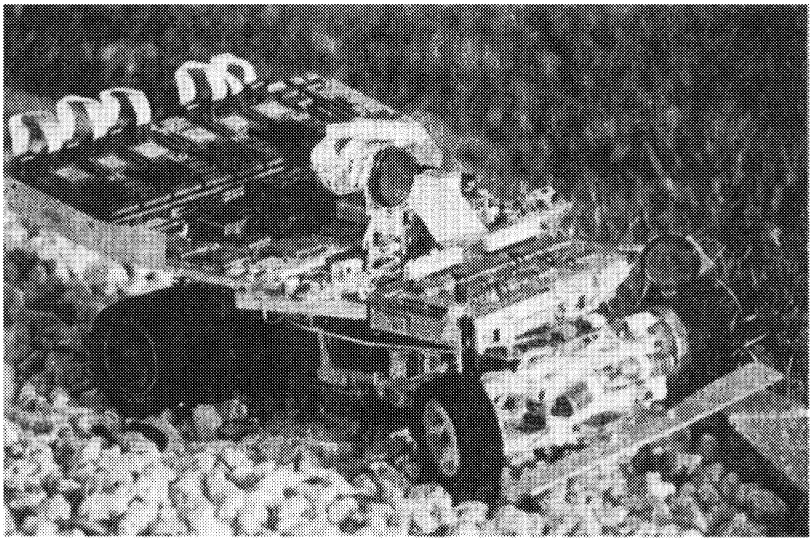


Figure 11–16. Fuzzy model car [von Altrock et al. 1992, p. 42].

car has a one-horsepower electric motor and can accelerate to 20 mph in 3.5 seconds. Furthermore it has advanced features such as individual wheel suspension, disk brakes, and differential and shock absorbers. Three polaroid sensors are used for orientation (front, left, and right), and additional infrared sensors are mounted in each wheel to measure the individual speed. The model car is shown in figure 11–16.

Since the conventional Mamdani max-min operators were not sufficient in this case, compensatory operators such as the γ-operator were used (see chapter 3). Another modification was the introduction of "rule weights" that are used to describe the plausibility of each rule. The objective of the car was to reach a target as fast as possible without hitting the walls or any obstacle. A typical experimental design is depicted in figure 11–17.

Most of the results were very encouraging. However, in some situations the car lost its orientation due to the limited information obtained from the sensors. This can only be avoided if some sort of memory is used to compute the current orientation [cf. von Altrock et al. 1992, p. 48].

11.7.3 Control of a Diesel Engine

Murayama et al. [1985] designed a fuzzy controller for a marine diesel engine. The objective here was to minimize the fuel consumption rate (FCR). The engine

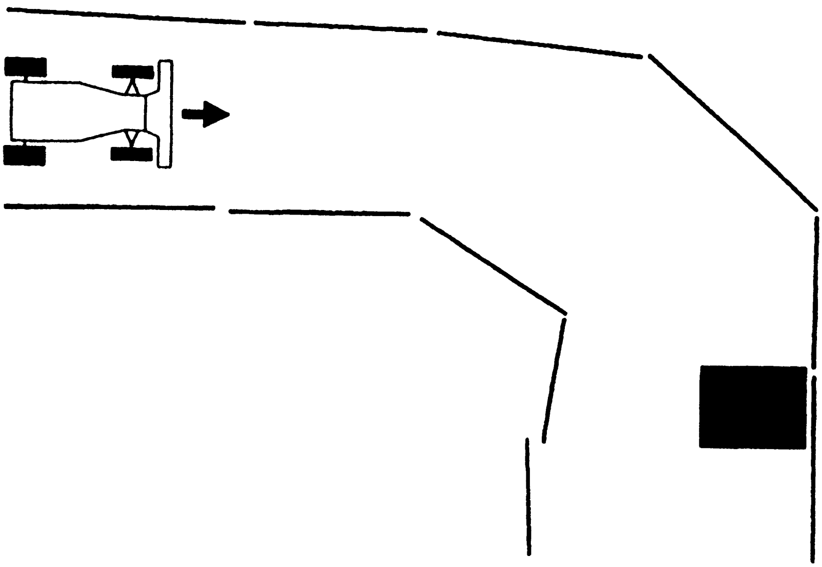


Figure 11–17. Experimental design [von Altrock et al. 1992, p. 48].

is controlled by fuel flow rate (*Q*), fuel injection timing (*U*), fuel injection duration (*T*), and inner pressure of the fuel pipe (*P*). Special attention was paid to the fuel injection timing, which influences the FCR directly. Figure 11–18 depicts the FCR as a function of the fuel injection timing.

Since the data are noisy, gradient methods cannot be employed directly. Therefore the authors use an adaptive method to verify the results obtained by the gradient search. Fuzzy numbers and an adjustment method that uses a fuzzy set to assess the credibility of the computed results are employed. The control algorithm is depicted in figure 11–19.

No rules are used to calculate the actual control output as in the Mamdani and the Sugeno controller. Therefore one may also consider this application as an application of fuzzy data analysis to a control problem. The results that were obtained with this simple method were, however, very encouraging. The fuzzy control method outperformed the conventional method clearly, as is shown in figure 11–20.

11.7.4 Fuzzy Control of a Cement Kiln

In this case, we will consider a physical process as the object of control. Let us first describe briefly the process itself [King and Karonis 1988, pp. 323].

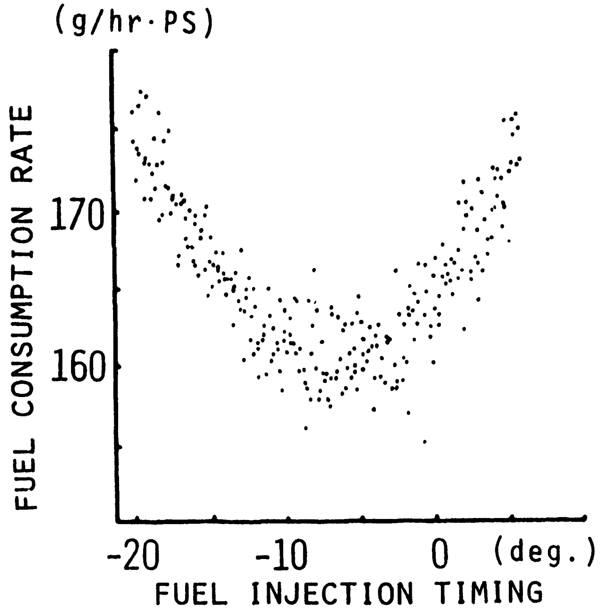


Figure 11–18. FCR vs. fuel injection timing [Murayama et al. 1985, p. 64].

Cement is manufactured by heating a slurry consisting of clay, limestone, sand, and iron ore to a temperature that will permit the formation of the complex compounds of cement, dicalcium silicate (C_2S), tricalcium silicate (C_3S), tricalcium aluminate (C_3Al), and tetracalcium aluminoferrite (C_4AlF). In the first stage of the kilning process, the slurry is dried and excess water is driven off. In the second stage, calcining takes place, with the calcium carbonate decomposing to calcium oxide and carbon dioxide. In the final stage, burning takes place at 1,250–1,450°C, and free lime (CaO) combines with the other ingredients to form the cement compounds. The end product of the burning process is referred to as clinker.

The kiln consists of a long steel shell about 130 m in length and 5 m in diameter. The shell is mounted at a slight inclination to the horizontal, and is lined with fire bricks. The shell rotates slowly, at approximately 1 rev/min, and the slurry is fed in at the upper or back end of the kiln. The inclination of the shell and its rotation transports the material through the kiln in about 3 hours 15 minutes with a further 45 minutes spent in the clinker cooler.

The heat in the kiln is provided by pulverized coal mixed with air, referred to as primary air. The hot combustion gases are sucked through the kiln by an induction fan at the back end of the kiln [Umbers and King 1981, p. 370].

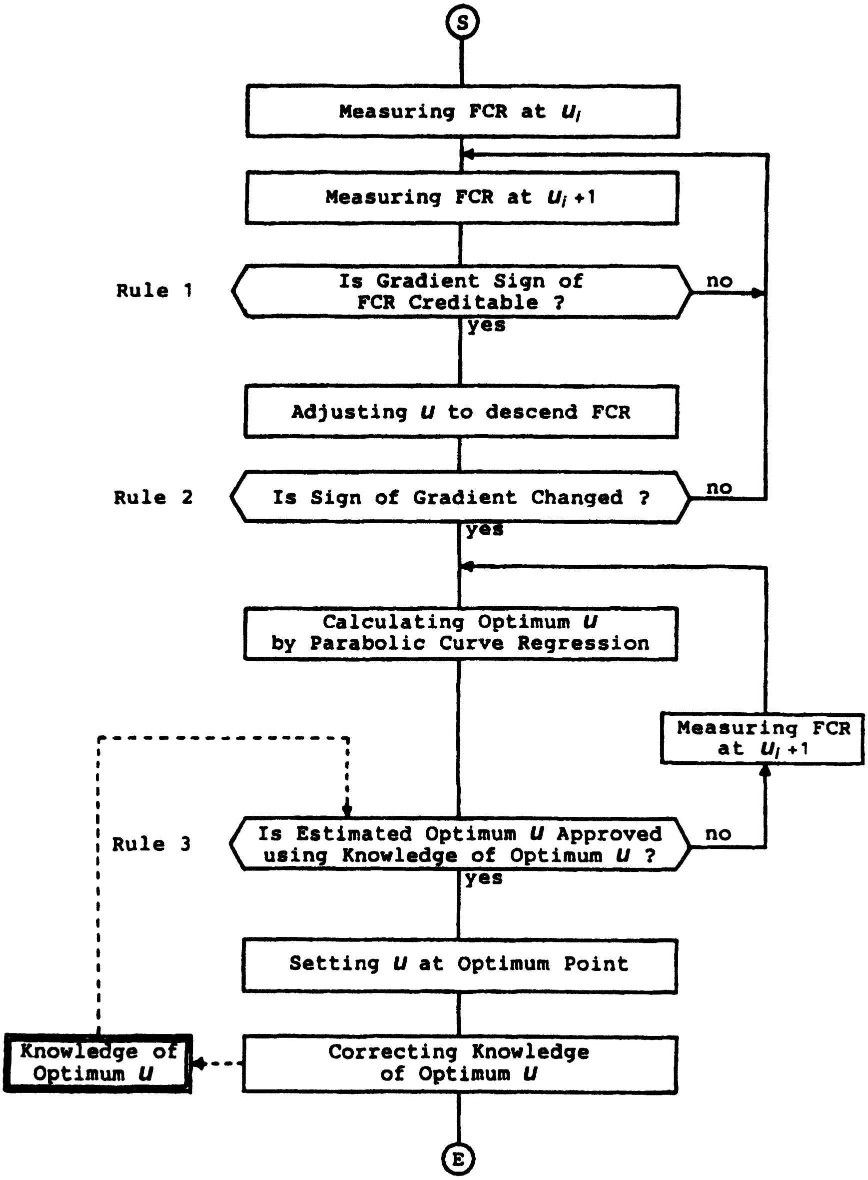


Figure 11–19. Control algorithm [Murayama et al. 1985].

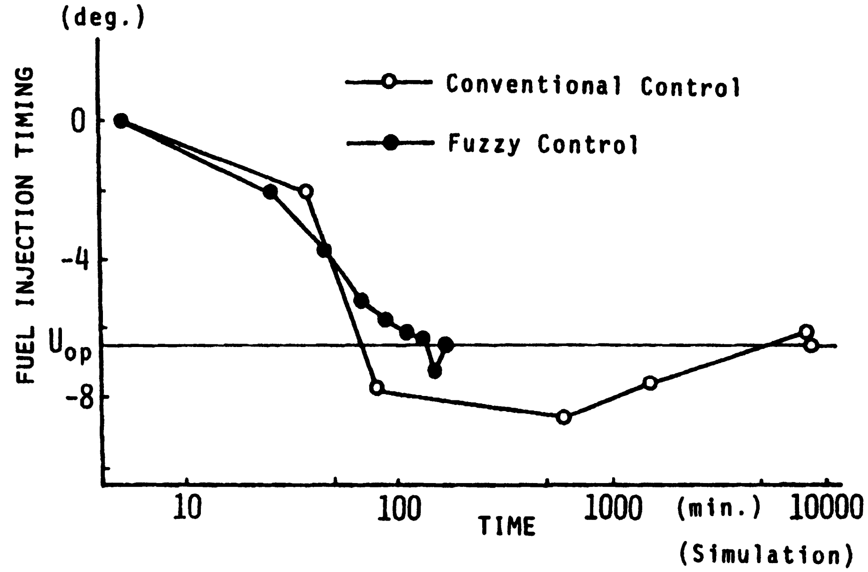


Figure 11–20. Experimental results [Murayama et al. 1985].

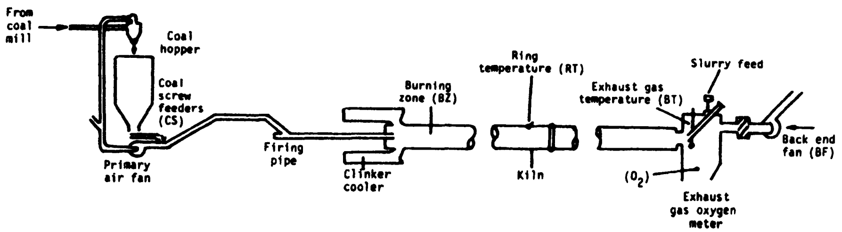


Figure 11–21. Schematic diagram of rotary cement kiln [Umbers and King 1981, p. 371].

Figure 11–21 illustrates the production process. The main problem in mathematically modeling a control strategy is that the relationships between input variables (measured characteristics of the process) and control variables are complex and nonlinear and contain time lags and inter-relationships; in addition, the kiln's response to control inputs depends on the prevailing kiln conditions. These were certainly reasons why a fuzzy control system was designed and used—which eventually even led to a commercially available fuzzy controller.

From the many possible input and control variables, the following were chosen as particularly relevant. Input variables include

1. exhaust gas temperature—back-end temperature (BT);
2. intermediate gas temperature—ring temperature (RT);
3. burning-zone temperature (BZ);
4. oxygen percentage in exhaust gases (O_2); and
5. liter weight (LW)—indicates clinker quality.

The process is controlled by varying the following control variables:

1. kiln process (KS);
2. coal feed (CS)—fuel; and
3. induced draught-fan speed (BF).

The calculation of the control action was composed of the following four stages:

1. calculate the present error and its rate of change;
2. convert the error values to fuzzy variables;
3. evaluate the decision rules using the compositional rule of inference; and
4. calculate the deterministic input required to regulate the process.

Concerning the control strategies used, let us quote Larsen:

> The aim of the computerized kiln control system is to automate the routine control strategy of an experienced kiln operator. The applied strategies are based on detailed studies of the process operator experiences which include a qualitative model of influences of the control variables on the measured variables [Larsen 1981, p. 337].

1. If the coal-feed rate is increased, the kiln drive load and the temperature in the smoke chamber will increase, while the oxygen percentage and the free lime content will decrease.
2. If the air flow is increased, the temperature in the smoke chamber and the free lime content will increase, while the kiln drive load and the oxygen percentage will decrease.

On the basis of thorough discussions with the operators, Jensen [1976] defined 75 operating conditions as fuzzy conditional statements of the type:

IF	drive load gradient is	(DL,SL,OK,SH,DH)
AND	drive load is	(DL,SL,OK,SH,DH)
AND	smoke chamber temperature is	(L,OK,H)
THEN	change oxygen percentage is	(VN,N,SN,ZN,OK,ZP,SP,P,VP)
PLUS	change air flow is	(VN,N,SN,ZN,OK,ZP,SP,P,VP)

The following fuzzy *primary terms* are used for the measured variables:

1. DL = drastically low	5. SH = slightly high
2. L = low	6. H = high
3. SL = slightly low	7. DH = drastically high
4. OK = ok	

The following fuzzy *primary terms* are used for the control variables:

1. VN = very negative	6. ZP = zero positive
2. N = negative	7. SP = small positive
3. SN = small nagative	8. P = positive
4. ZN = zero negative	9. VP = very positive
5. OK = ok	

The *linguistic terms* are represented by membership functions with four discrete values in the interval [0, 1] associated with 15 discrete values of the scaled variables in the interval [−1, +1].

In order to simplify the implementation of the fuzzy logic controller, Ostergaard [1977] defined 13 *operating conditions* as fuzzy conditional statements of the type:

IF	drive load gradient is	(SN,ZE,SP)
AND	drive load is	(LN,LP)
AND	free lime content	(LO,OK,HI)
THEN	change burning zone temperature	(LN,MN,SN,ZE,SP,MP,LP)

The following fuzzy *primary terms* are used:

1. LP = large positive	7. SN = small negative
2. MP = medium positive	8. MN = medium negative
3. SP = small positive	9. LN = large negative
4. ZP = zero positive	10. HI = high
5. ZE = zero	11. OK = ok
6. ZN = zero negative	12. LO = low

The 13 operating conditions are defined by taking only some of the combinations into account, and by including also the previous values of the drive load gradient, the latter being calculated from the changes in the drive load. In order to decide whether the oxygen percentage set point or the air flow should be changed, three additional fuzzy rules for each operating condition are formulated based on the actual values of the oxygen percentage and the smoke chamber temperature, resulting in 39 control rules.

Details of membership functions used can be found in Holmblad and Ostergaard [1982] and results of testing the system in Umbers and King [1981] and Larsen [1981]. We shall not describe these details here, primarily because they are not of high general interest.

Before we turn to a quite different type of control, it should be mentioned, however, that the reader can find descriptions and references to more than 10 further projects of the type described here in Mamdani [1981], in Pun [1977], and in Sugeno [1985a].

11.8 Tools

Fast and easy implementation of control systems requires adequate tools that assist the system designer in the design and coding, which would be time-consuming if performed by hand. An increasing number of tools exist both for conventional and fuzzy logic control. Modern tools use graphical animation and offer interactive on-line development capabilities instead of precompiling. Precompiler tools precompile the linguistically designed controller into a code, e.g., in C. This can then be combined with other codes. Then the controller is started and the closed-loop behavior is observed. If the behavior isn't sufficient—which usually is the case—the control is interrupted and a new, modified, controller is defined and precompiled. This controller is linked to the process and so on. This method is inefficient and time-consuming, since every modification implies interruption of the control and compiling and linking.

The interactive approach is much more efficient because the designer is enabled to study the direct consequences of modifications of design parameters such as rules and fuzzy sets. Here we shall consider, as an example, the *fuzzy* TECH design tool by INFORM [Inform 1995]. Figure 11–22 shows the development philosophy of *fuzzy* TECH.

This tool runs on most hardware platforms and can be used for on-line optimization of a fuzzy control system.

The system introduces the concept of "normalized rule bases" that makes even large rule bases easy to comprehend. A screenshot of a rule base for the model car [von Altrock et al. 1992] is shown in figure 11–23.

The whole inference process is visualized in different windows on-line, and auxiliary screens visualizing the phase plane and transfer characteristics help the designer in tracing erroneous rules or term definitions. Figure 11–24 shows the simulation screen of the model car presented at the FUZZ-IEEE conference in 1992.

We sum up by stating that FLC design is accelerated and made more efficient by the use of modern graphical development tools. Such tools can also be used effectively for training purposes in connection with simulation models or laboratory processes.

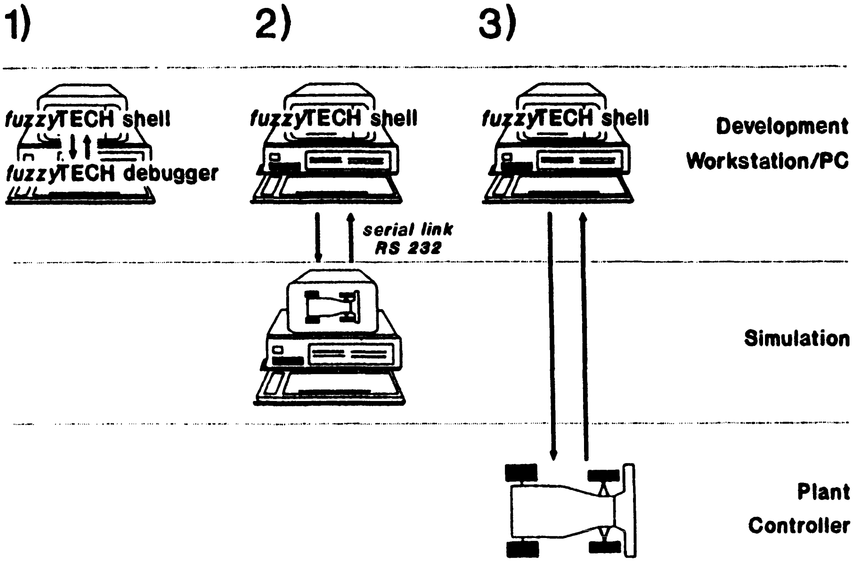


Figure 11–22. Controller development in *fuzzy*TECH [von Altrock et al. 1992].

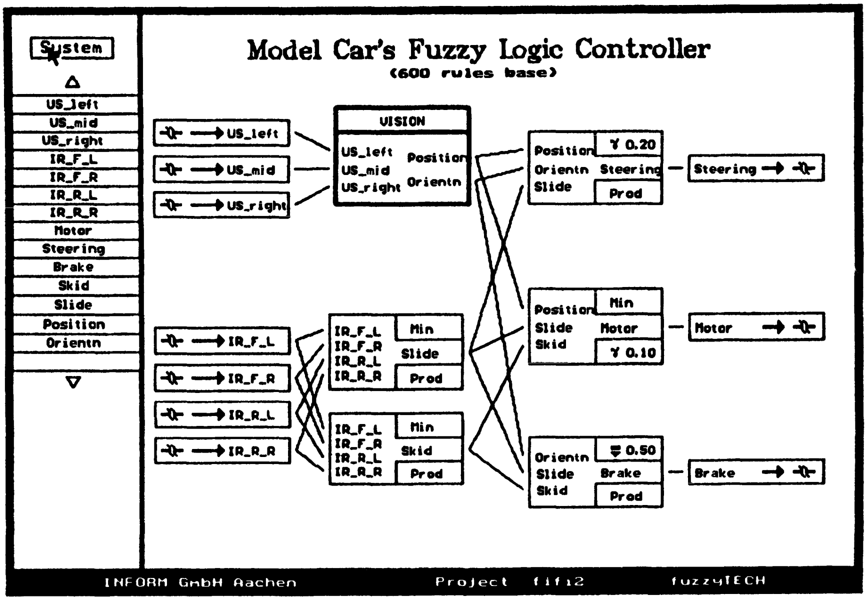


Figure 11–23. Rule base for model car [von Altrock et al. 1992].

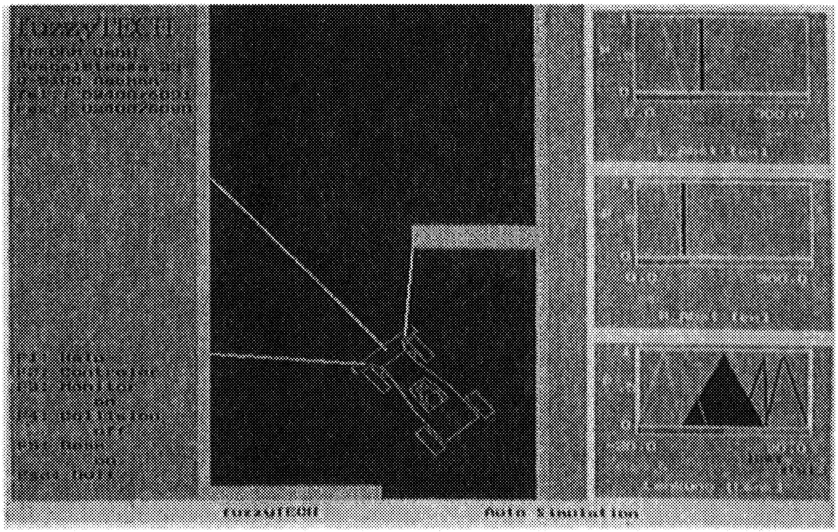


Figure 11–24. Simulation screen [von Altrock et al. 1992].

11.9 Stability

Stability and performance of the closed-loop system are considered by many control engineers to be the main criteria assessing the quality of a control system. In many cases it is desirable to prove the stability of the controlled system. It is, of course, only possible to prove the stability of the process *model* and not of the process itself; however, stability can often be proved for a wide range of model parameters, and the risk of instability can thus be minimized. The lack of formal techniques for stability analysis has been a main point of criticism of FLC systems. There do, however, already exist many approaches to prove the stability of a closed-loop FLC system.

When studying the stability of FLC systems, one has to use a model of the process that can be fuzzy or crisp. Most methods use crisp process models and conventional nonlinear control theory to prove stability. In this context, the fuzzy controller is considered as a nonlinear transfer element, i.e., the output is determined as a function of the input variables, $u = \Phi(r)$ [Kickert and Mamdani 1978]. Such a system is depicted in figure 11–25. Set-point values and noise can be neglected because stability is a system property. This means that the control action for a known input value can be derived by calculating the result of rule firing,

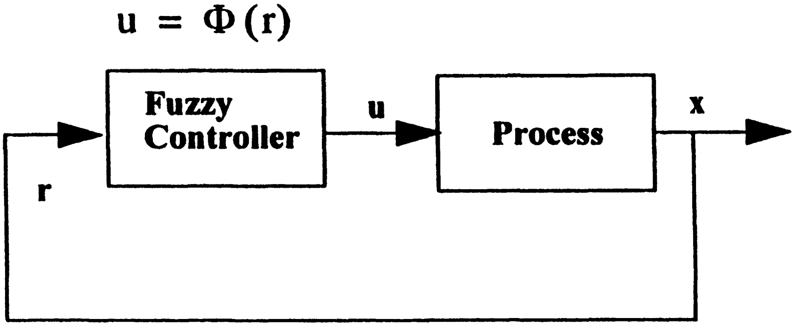


Figure 11–25. Fuzzy controller as a nonlinear transfer element.

rule aggregation, and defuzzification. The problem is often to find a suitable representation of the fuzzy controller in this context.

In the case of a nonlinear crisp process model, one can distinguish between time-domain and frequency-domain models [Bretthauer and Opitz 1994]. The time-domain models include the state-space approach, Ljapunov theory, hyperstability theory, and the bifurcation theory approach. The class of frequency-domain methods include the harmonic-balance approach and the circle and Popov criteria. Figure 11–26 summarizes the different approaches.

A graphical approach to stability analysis is the state-space approach, where the trajectory of the closed-loop system is displayed in the two-dimensional state space. Naturally, this approach is limited to two-dimensional systems. The main idea is to partition the space that is defined by the input base variables of the rules, which is called the *linguistic state space*, according to the terms of the linguistic variables. This leads to sections of the state space where the degree of membership of an input variable x_i in one term—say, term k—is higher than the degree of membership in the other terms, i.e., $\mu_i^{k_i}(x_i) \geq \mu_i^{j_i}(x_i)$ for all $j_i \neq k_i$. Since the rule base was defined in terms of these input variables (see table 11–1), we can infer which term of the output variable is dominant in the corresponding sector of the state space. Figure 11–27 shows the linguistic state space that corresponds to our heating system example. Note that every input consisting of a temperature and a change of temperature can be located in the state space.

Suppose that we start the controller with an input temperature of 13 degrees and a change of temperature of $-1°$ per minute. The controller starts the heating system with approximately medium power, and the temperature rises. Due to this control action, other regions of the state space are reached and other rules get dominant. The sequence of regions that are reached in the state space depends on

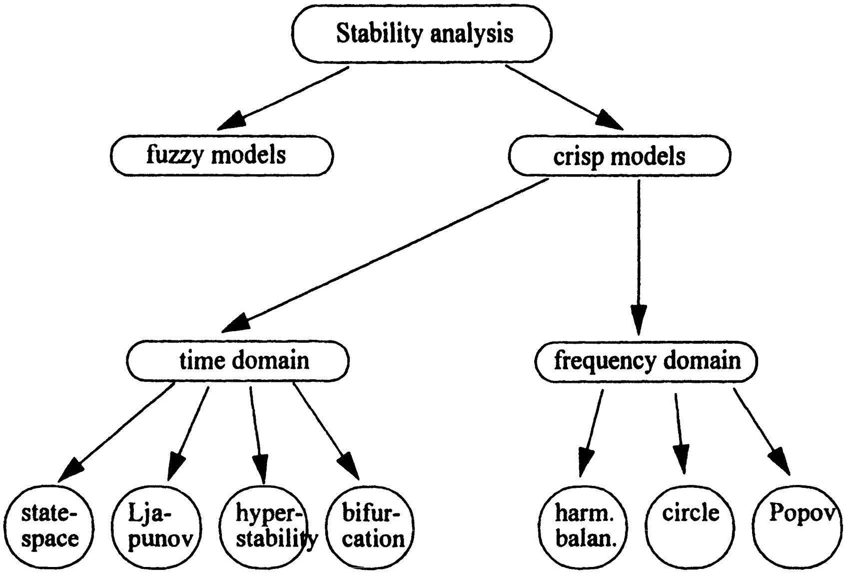


Figure 11–26. Classification of stability analysis approaches.

the fuzzy controller and the system to be controlled and is called the *linguistic trajectory*. A possible linguistic trajectory of the heating system example is depicted in figure 11–28. The corresponding linguistic trajectory is (l,nb), (l,ns),(l,z),(l,ps),(c,ps),(h,ps),(h,z),(c,z) where the first entry is the term of the linguistic variable temperature and the second entry is the term of the linguistic variable change of temperature, e.g., (l,ns) means the region with low temperature and negatively small change of temperature. The linguistic trajectory shows that the system reaches an equilibrium point, namely, (c,z), where the temperature is comfortable and does not change. If an equilibrium point is reached for all possible starting configurations in the state space, then the system is stable. The state space approach has the advantage of being easy to understand and is of great help when designing a fuzzy controller, since the impact of rules can be seen directly in the state space. Some software tools offer the possibility of plotting the linguistic trajectory of the system on the computer screen. We close the discussion of this approach by noting that a system that reaches an equilibrium point in the linguistic state space may have underlying oscillations which cannot be detected by this method due to the coarseness of the partition induced by the membership functions of the terms of the linguistic variables. The heating system may, as an example, lead to temperatures varying between 18° and 19° Celsius

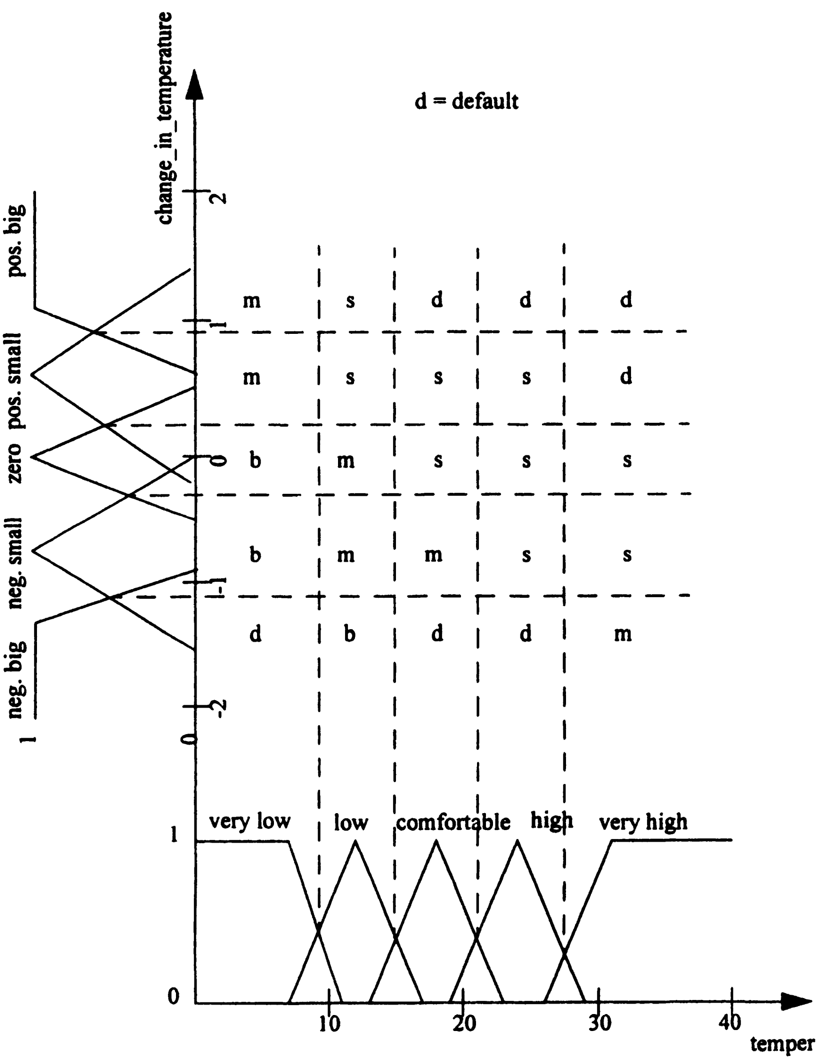


Figure 11–27. Linguistic state space.

and small negative and positive changes of temperature if the power can only be adjusted discretely. The activated region in the state space would, however, always be (c,z).

Since the introduction of the formal methods of FLC stability analysis requires a solid background in nonlinear control theory, a detailed discussion of the

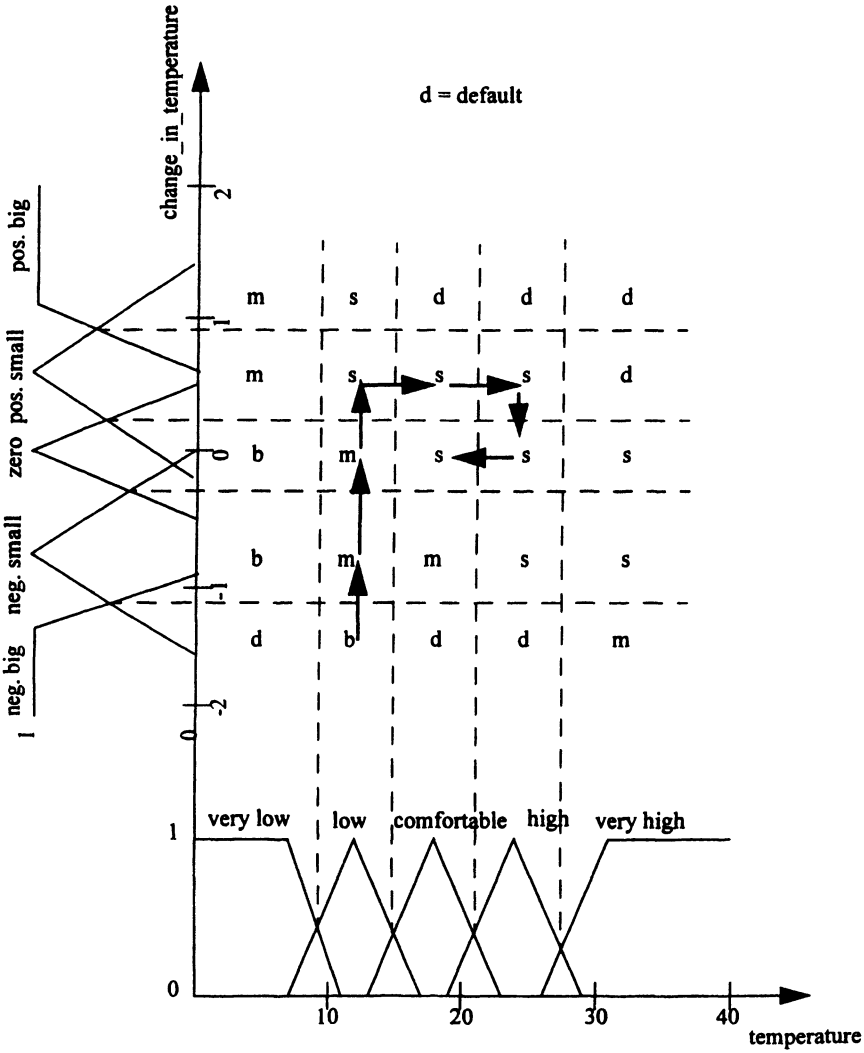


Figure 11–28. Linguistic trajectory.

approaches is not possible in this book. We limit ourselves to the specification of the different approaches and request interested readers to consult the literature. Topics and references include the following: controller as relay [Kickert and Mamdani 1978], limit theorems [Bouslama and Ichikawa 1992], fuzzy sliding mode control [Hwang and Lin 1992], Ljapunov theory [Langari and Tomizuka

1990; Tanaka and Sugeno 1992; Kiendl and Rüger 1993], harmonic balance [Kiendl and Rüger 1993], circle criterion [Ray and Majunder 1984], conicity criterion [Aracil et al. 1991], and vector fields [Aracil et al. 1988, 1989].

An overview of some of these approaches is found in Driankov et al. [1993], and a literature survey is given by Bretthauer and Opitz [1994].

11.10 Extensions

Most of the basic problems of FLC have been resolved, and researchers are now investigating advanced topics such as stability, adaptive fuzzy control, hybrid systems, neuro-fuzzy systems, and FLC systems tuned by genetic algorithms (GAs) that are inherently adaptive systems. Progress is fast in these areas, and promising experimental results have been obtained.

With the rising popularity of FLC, more engineers will be trained in this area in the future. This training will lead to more applications of FLC systems and to rising field experience of the involved engineers. Fuzzy logic control is an integral part of modern control theory, not replacing conventional methods but rather complementing them.

Since the literature in fuzzy control is too vast to be discussed in its entirety in this textbook, a summary is given below. It is primarily intended for those who have an extended interest in this area:

One of the first books on fuzzy logic control was written by W. Pedrycz in 1989 [Pedrycz 1989] and focuses on many concepts of FLC. The use of fuzzy relations in connection with FLC systems is discussed thoroughly. A second edition of this popular book appeared in 1993 [Pedrycz 1993] and covers also new directions, such as neural network methods. Many survey articles on FLC have appeared in control journals in the last years, and we very much recommend the survey of Lee [1990], which covers all basic aspects. The first major book on applications was the one edited by Sugeno [1985a]. Zimmermann and von. Altrock [1994] provide a more recent collection of applications, most of them describing German industrial projects. Jamshidi et al. [1993] also cover a wide area of different applications, including robotics and flight control, most of which have been realized in the United States. An interesting collection of the now-famous Japanese applications of fuzzy control is provided by Hirota [1993]. Many articles do describe practical implementations of FLC systems and can be found in journals covering mainly fuzzy sets as well as in journals on automatic control. From an engineer's point of view, the book written by Driankov, Hellendoorn, and Reinfrank [1993] covers all major aspects of fuzzy control. A background in conventional control theory is, however, necessary to understand some of the chapters.

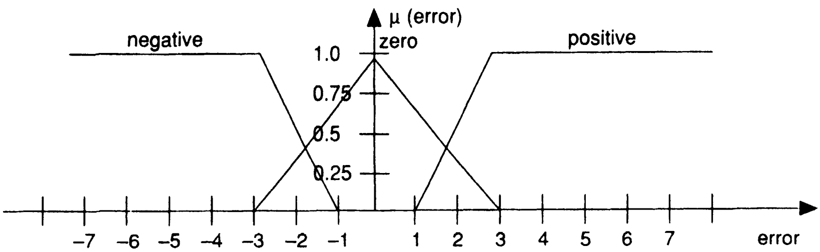
Exercises

1. a. Draw the block diagram of a Mamdani/Sugeno controller and explain each function separately.
   b. What are the differences between the Mamdani and the Sugeno controller?
2. Which design parameters can be varied in a fuzzy controller?
3. A Mamdani controller has the following rule base:

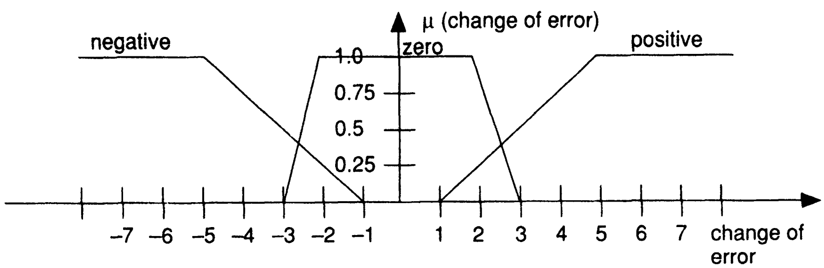
error/change of error	negative	zero	positive
negative	big		
zero	big	medium	medium
positive		small	small

The linguistic variables are defined as follows:

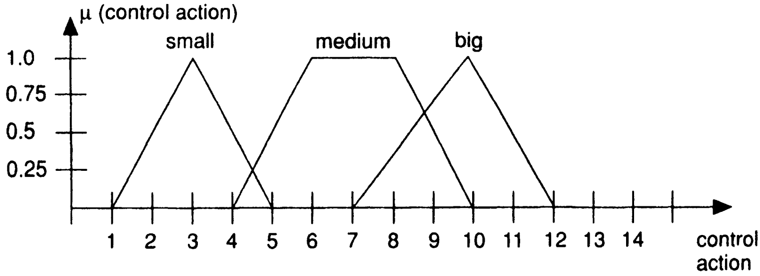
Error:



Change of error:



Control action:



a. Calculate the fuzzy set of control, when error = 2 and change of error = 4.
b. Calculate the control action when
   (i) mean of maxima
   (ii) center of sums

   is used as a defuzzification procedure.

4. Which operators can be varied in the Mamdani controller? Discuss the choice of operators in connection with fuzzy controllers.

12 FUZZY DATA BASES AND QUERIES

12.1 Introduction

Data bases are one form of modeling parts of the real world. They may contain descriptions of technical systems, of enterprises, of scientific activities, of landscapes (geographical information systems), or other domains. The world of data bases is the world of digital computers, one of the most typical dichotomous systems. It is, therefore, not surprising that the type of storage is crisp and that all data processing, e.g. input, storage, querying is crisp, no matter whether the factual relationships described in a database are crisp or uncertain or fuzzy.

For approximately 20 years researchers around the world have been concerned with the use of fuzzy set theory to represent imprecision in data bases. This research has been hampered by the fast development of data base technology. From the graphtheoretic paradigm data base theory moved to relational databanks and on to object-oriented designs, each of these paradigms requiring different fuzzy approaches. This is probably one of the reasons why commercial realizations of fuzzy databank technology lag behind the theory.

In this book and chapter we cannot describe all existing fuzzy approaches in fuzzy databank technology (interested readers are referred to [Petry 1996; Bordogna and Pasi 2000; Pons et al. 2000]). We shall rather focus our attention exemplarily on relational databanks and on similarity based fuzzy models.

12.2 Fuzzy Relational Databases

The relational data model is based on set-theoretic concepts. Essentially, relational data bases consist of relations in two-dimensional (row and column) format. Rows are called tuples and correspond to records and columns are called domains or attributes and correspond to fields. One or more attributes are distinguished as the key attributes. We will consider relations of the so-called "third normal form", which possess two characteristics: first, each attribute fully depends on the entire key (and not part of it). Secondly, each of the non-key attributes is non-transitively dependent on the key (i.e. they depend only on the key and not on each other).

Example 12–1

Let us consider a data base that describes materials which are supplied by different suppliers. The first table shows the suppliers together with their locations, the material supplied and their evaluated quality. The second table contains again the suppliers and information about their delivery reliability and their costs and the third table describes the materials supplied.

Suppliers

supplier	**location**	**material**	**quality**
DEWAG	Paris	802.025	medium
DEWAG	Paris	802.020	medium
MAM	Berlin	802.025	high
KBA	Hamburg	802.025	high
INFORM	Aachen	802.025	low

Reliability

supplier	material	reliability
DEWAG	802.025	high
DEWAG	802.020	medium
MAM	802.025	medium
KBA	802.025	low
INFORM	802.025	high

Materials

material	description	standard
802.020	engine XL	EURO
802.025	engine L	EURO
802.020	engine XL	ISO

Access to a database via a query is normally based on relational algebra. This allows to manipulate and combine the relations or tables that the requested query results are provided.

A relational algebra operation consists of an operation name, one or more relation names, one or more domain names and an optional conditional expression. For example, an operation on the above relations might be:

Select *Companies* where *Material* = EURO-NORM and Location = Paris

which would result in:

DEWAG in Paris.

#

So far all components in the relations were crisp. If this is not an adequate description of reality, fuzzy rather than crisp relations might be used (see chapter 6).

The fuzziness of such a relation can either be modeled by considering linguistic values of the domains of attributes as terms of linguistic variables (see chapter 9), or one can assign to the relations an additional degree of membership.

In this case the table "Reliability" in the last example would, for instance, look as follows:

Reliability

supplier	material	reliability	μ_R
DEWAG	802.025	high	.8
DEWAG	802.020	medium	.7
MAM	802.025	medium	.6
KBA	802.025	low	.8
INFORM	802.025	high	.9

In this case the "values" for the attribute reliability would obviously be considered as being crisply defined (as symbols) and the μ_R would indicate the degree to which the relation is true. There might be another table which shows the degrees of membership for other "reliabilities" (high, low or medium) of the suppliers.

Fuzzy data bases are still very seldom in practice. One of the reasons may be that companies are very hesitant to replace their (crisp) data based by fuzzy data banks before they are convinced that it is worthwhile or necessary to do this.

Another application of fuzzy set theory is to design fuzzy query languages to crisp data bases. This might avoid replacing existing crisp data banks and still taking advantages of the strength of fuzzy set theory.

12.3 Fuzzy Queries in Crisp Databases

With respect to databases fuzzy sets can primarily be used in two directions: first to differentiate between different degrees of relevance, strength of relations etc. Secondly, they can also be used to reduce complexity, i.e. to extract from large masses of data relevant information. The first goal was considered in the last section. Now we want to focus on the second goal.

In the last section of this chapter we called all the values that an attribute could have the domain of this attribute. From a user's point of view not all values in the domain of an attribute will have to be considered different. Values may be distinguishable, i.e. 4 and 5, but the user might consider them as indifferent in the context of a certain query.

We shall call elements in the domain of an attribute that have, in a certain context, the same meaning "equivalent". This can be expressed in the form of an equivalence relation.

Example 12–2

The domain of the attribute "quantity" be defined as

$$D_q = \{\text{high, medium, sufficient, low}\}.$$

For the purpose of a certain query the user is only interested whether the quality is either "high or medium" or "sufficient or low".

This can be expressed by the following equivalence relation, E:

E	**high**	**medium**	**sufficient**	**low**
high	1	1		
medium	1	1		
sufficient			1	1
low			1	1

Hence, the domain of the attribute "quality" in this context is partitioned into two subsets of equivalent values which we will call "equivalence classes".

Expressed differently:

$$C(\text{quality}) = \{\{\text{high, medium}\}, \{\text{sufficient, low}\}\}.$$

In the context C the equivalence relation has partitioned the domain of the attribute quality into two equivalence classes.

The introduction of equivalence classes obviously reduces the complexity of the data to be considered by reducing the number of component of vector D_q to those of vector C.

Example 12–3 [Schindler 1997]

Let us consider the following data base which describes suppliers delivering materials with different qualities and different delivery delays:

supl.	supplier	material	quality	delay
	BAW	802.025	sufficient	8
	DEWAG	802.025	medium	5
	DEWAG	809.200	high	8
	KBA	802.025	sufficient	7
	KBA	809.200	sufficient	3
	KBA	840.024	low	9
	MD	802.025	sufficient	8
	MD	809.200	medium	4
	MTX	802.025	high	2
	MTX	840.024	high	4
	MAM	802.025	low	7
	MAM	840.024	medium	6
	ZT	809.200	high	8
	ZT	840.024	medium	2

The domains of "quality" and "delay" are

$$D_q = \{\text{high, medium, sufficient, low}\}.$$

$$D_d = [1, 10].$$

The goal of a query is to evaluate the suppliers in 4 groups, such that appropriate measures can be taken to improve the supply situation.

The manager of the purchasing department believes that for the query the following contexts are appropriate:

$$C_q(\text{quality}) = \{\{\text{high, medium}\}, \{\text{sufficient, low}\}\}.$$

{high, medium} is considered good quality and {sufficient, low} indicates bad quality.

$$C_d(\text{delay}) = \{[1, 5], [5, 10]\},$$

where a delay of [1, 5] is considered acceptable and (5, 10] is considered unacceptable.

Applying these contexts the our data base we obtain the following classes:

supl.	**supplier**	**quality**	**delay**	
	{MTX, DEWAG, MD, ZT}	{high, medium}	{2, 4, 5}	C1
	{DEWAG, ZT, MAM}	{high, medium}	{6, 8}	C2
	{KBA}	{sufficient}	{3}	C3
	{BAW, MD, KBA, MAM}	{sufficient, low}	{7, 8, 9}	C4

An interpretation of these 4 classes is shown in the following matrix:

<table>
<tr><td rowspan="5">unacceptable</td><td>10</td><td rowspan="5" colspan="2">C2
ask supplier to
decrease delays</td><td rowspan="5" colspan="2">C4
terminate
relationship</td><td rowspan="5"></td></tr>
<tr><td>9</td></tr>
<tr><td>8</td></tr>
<tr><td>7</td></tr>
<tr><td>6</td></tr>
<tr><td rowspan="5">acceptable</td><td>5</td><td rowspan="5" colspan="2">C1
expand
relationship</td><td rowspan="5" colspan="2">C3
ask supplier to
improve quality</td><td rowspan="5"></td></tr>
<tr><td>4</td></tr>
<tr><td>3</td></tr>
<tr><td>2</td></tr>
<tr><td>1</td></tr>
<tr><td colspan="2"></td><td>high</td><td>medium</td><td>suff.</td><td>low</td><td>D (quality)</td></tr>
<tr><td colspan="2"></td><td colspan="2">good</td><td colspan="2">bad</td><td>quality</td></tr>
</table>

Obviously suppliers in one class are not distinguishable according to their attractiveness. This might be demotivating for suppliers when they improve quality or delay and still remain in the same class. One way to improve this situation is to define fuzzy sets over the attributes "quality" and "delay".

Let us define the following two linguistic variables:

The linguistic variable "delay" shall have two terms "acceptable" and "unacceptable" with the following membership functions:

$$\mu_{acc.}(u) = \begin{cases} 1 & for \quad 1 \le u < 3 \\ (7-u)/(7-3) & for \quad 3 \le u < 7 \\ 0 & for \quad 7 \ge u \end{cases}$$

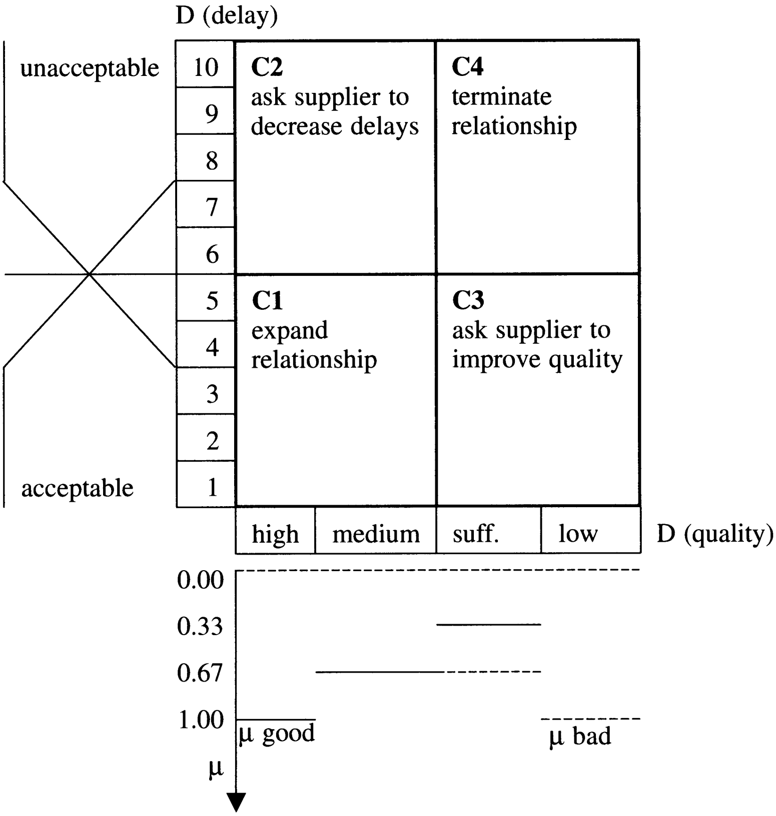
$$\mu_{unacc.}(u) = \begin{cases} 0 & for \quad u < 3 \\ (u-3)/(7-3) & for \quad 3 \le u < 7 \\ 1 & for \quad 7 \le u \end{cases}$$

For the linguistic variable "delay" we shall define the two terms "good" and "bad" with the following membership functions:

$$\mu_{\text{good}}(u) = \{(\text{high}, 1), (\text{medium}, .67), (\text{sufficient}, .33)\}$$

$$\mu_{\text{bad}}(u) = \{(\text{medium}, .33), (\text{sufficient}, .67), (\text{low}, 1)\}.$$

Graphically the class matrix would now book as follows:



For the various suppliers the degrees of membership in the different terms can easily be determined by substituting the values of the attributes in the membership functions.

The supplier "BAW" in the data base, for instance, would provide material which is of good quality to the degree .33 and of bad quality to the degree .67. His delay is acceptable to the degree 0 and unacceptable to the degree 1.

We might, however, also be interested in either the degree of membership to which either a supplier belongs to the various classes or the degree to which he is "attractive", where "attractive" can be considered as "having good quality *and* an acceptably delivery delay".

In their case we have to aggregate the respective degrees of membership. Since it is an "and" aggregation, we could either use a t-norm or a compensatory aggregation. We shall assume that the two attributes are compensatory and, therefore, choose the "compensatory and" (definition 3–20). We shall compute the degree of membership of the suppliers in the different classes and use $\gamma = .5$.

For supplier "BAW" the degrees of membership for classes 1 and 3 are obviously 0.

For class 2 the terms "unacceptable" (of delay) and "good" (of quality) are relevant. For "BAW" these are 1 (delay of 8) and .33 (sufficient quality), respectively.

Hence, using the γ-operator with $\gamma = .5$:

$$\begin{aligned}\mu_{c2}(BAW) &= (1 \cdot .33)^{.5}(1-(1-.33)(1-0))^{.5}\\ &= .57 \cdot 1 = .57\end{aligned}$$

For class 4 we would obtain accordingly $\mu_{c4}(BAW) = .82$.

Obviously these two degrees of membership do not add up to 1. If we want to obtain normalized degrees of membership, we can divide all degrees of membership for "BAW" by the sum of degrees of membership of "BAW" to the 4 classes (this is the cardinality according to definition 2–5).

For "BAW" the cardinality is (.57 + .82) = 1.39 and hence, we obtain the class memberships of

$$\begin{aligned}\mu_{BAW}(1) &= 0\\ \mu_{BAW}(2) &= .41\\ \mu_{BAW}(3) &= 0\\ \mu_{BAW}(4) &= .59\end{aligned}$$

The remaining suppliers supply more than one material.

Here we have two alternative ways of proceeding, depending on whether we are interested in a specific material delivered by several suppliers or whether we want to evaluate suppliers with respect to all materials they supply. We will assume the latter. In this case we compute the degrees of membership for all materials, suppliers and classes separately and then add the degrees of membership of different materials of one supplier for each class.

"MTX", for example, supplies two materials (802.025 and 840.024) with different ratings. Let us consider class 1:

Material "802.025" has degrees of membership of 1 and 1 respectively. Material "840.025" has 1 and 0.75. Hence, the first material has an (unnormalized) degree of membership of 1 and "840.024" one of .87.

Following Ozawa and Yamada [1994] we add these two degrees of membership to determine the degree of membership of MTX to class 1. After we have determined the (unnormalized) degrees of membership of MTX to the other classes we will find that the cardinality for MTX is 2.37. Hence: MTX belongs to class 1 to the degree

$$\frac{1.87}{2.37} = .79.$$

The following table shows the unnormalized degrees of membership of the suppliers to the classes. The last row of this matrix shows the respective cardinalities. If these are used for normalization, we arrive at the subsequent matrix of normalized degrees of membership.

	BAW	**DEWAG**	**KBA**	**MD**	**MTX**	**MAM**	**ZT**	
C1	0.00	0.53	0.57	0.68	1.87	0.36	0.82	expand relationship
C2	0.57	1.53	0.57	0.93	0.50	0.68	1.00	ask supplier to decrease
C3	0.00	0.33	0.82	0.45	0.00	0.20	0.57	delays ask supplier to
C4	0.82	0.33	1.82	1.02	0.00	1.45	0.00	improve quality
	1.39	2.72	3.78	3.08	2.37	2.69	2.39	terminate relationship

Partition matrix of non-normalized degrees of membership

	BAW	DEWAG	KBA	MD	MTX	MAM	ZT	
C1	0.00	0.20	0.15	0.22	0.79	0.13	0.34	expand relationship
C2	0.41	0.56	0.15	0.30	0.21	0.25	0.42	ask supplier to decrease
C3	0.00	0.12	0.22	0.15	0.00	0.07	0.24	delays ask supplier to
C4	0.59	0.12	0.48	0.33	0.00	0.55	0.00	improve quality
	1.00	1.00	1.00	1.00	1.00	1.00	1.00	terminate relationship

Partition matrix of normalized degrees of membership

In above example the aggregation of the degrees of membership was performed by using the γ-operator with $\gamma = 0.5$. As was already described in chapter 3, this models an aggregation in the middle of the "logical and" and the "liguistic or". More or less compensation can be achieved by varying the γ between zero and one. It might also be appropriate to assign different weights (importance) to the various attributes. This is also possible when using the γ-operator, requires some caution, however (see [Zimmermann and Zysno 1983]).

13 FUZZY DATA ANALYSIS

13.1 Introduction

The terms *data analysis, pattern recognition*, and *data mining* are often used synonymously, and we shall do the same here. On the one hand, this area is one of the oldest and most obvious application areas for fuzzy set theory. On the other hand, pattern recognition existed long before fuzzy sets became known.

This topic embraces a very large and diversified literature. It includes research in the areas of artificial intelligence, interactive graphic computers, computer aided design, psychological and biological pattern recognition, linguistic and structural pattern recognition, and a variety of other research topics. One could possibly distinguish between mathematical pattern recognition (primarily cluster analysis) and nonmathematical pattern recognition. One of the major differences between these two areas is that the latter is far more context dependent than the former: a heuristic computer program that is able to select features of chromosomal abnormalities according to a physician's experience will have little use for the selection of wheat fields from a photo-interpretation viewpoint. In contrast to this example, a well-designed cluster algorithm will be applicable to a large variety of problems from many different areas. The problems will again be different for structural pattern recognition—when, for instance, handwritten H's should be distinguished from handwritten A's, and so on.

Verhagen [1975] presents a survey of definitions of pattern recognition that also cites the difficulties of any attempt to define this area properly. Bezdek [1981, p. 1] defines pattern recognition simply as "A search for structure in data."

The most effective search procedure—in those instances in which it is applicable—is still the "eyeball" technique applied by human "searchers." Their limitations, however, are strong in some directions: Whenever the dimensionality of the volume of data exceeds a limit, and the human senses, especially the vision, are not able to recognize data or features, the "eyeball" technique cannot be applied.

One of the advantages of human search techniques is the ability to recognize and classify patterns in a nondichotomous way. One way to imitate this strength is the development of statistical methods in mathematical pattern recognition, which in connection with high-speed computers have shown very impressive results. There are data structures, however, that are not probabilistic in nature or not even approximately stochastic. Given the power of existing EDP, it seems very appropriate and promising to find nonprobabilistic, nondichotomous models and structures that enable us to recognize and transmit in a usable form patterns of this type, which humans cannot find without the help of more powerful methods than "eyeball-search." Here, obviously, fuzzy set theory offers some promise. Fuzzy set theory has already been successfully applied in different areas of pattern search and at different stages of the search process. In the references, we cite cases of linguistic pattern search, of character recognition [Chatterji 1982], of visual scene description [Jain and Nagel 1977], and of texture classification [Hajnal and Koczy 1982]. We also give references for the application of fuzzy pattern recognition to medical diagnosis [Fordon and Bezdek 1979; Sanchez et al. 1982], to earthquake engineering [Fu et al. 1982], and to pattern search in demand [Carlucci and Donati 1977].

Another way to describe the main goal of data analysis is complexity reduction, in the sense that data masses that cannot be comprehended by human beings are reduced to lower-dimensional information that can be used, for instance, by human decision makers to support their decisions.

In data analysis, *objects* are considered that are described by some *attributes*. Objects can, for example, be persons, things (machines, products, . . .), time series, sensor signals, process states, and so on. The specific values of the attributes are the data to be analyzed. The overall goal is to find structure (information) about these data. This can be achieved by classifying the huge amount of data into relatively few classes of similar objects. This leads to a complexity reduction in the considered application, which allows for improved decisions based on the information gained.

The process of data analysis normally starts with the description of the process or the set of data that is to be analyzed. This process is very nontrivial, often least supported by tools, and generally leads to a high-dimensional model (one dimension corresponding to one property of the data or process). In feature analysis, the first reduction of complexity (dimension) is reached by reducing the number of properties to those that are most important, i.e., that contribute most to the description of the process or data set. Since this reduction is generally not yet sufficient, an additional reduction is achieved by defining in feature space a small number of classes. This stage is called classifier design, and it more or less terminates the preparatory steps of data analysis. These classes are now used, either in a batch type operation or continuously, to assign single objects or data to classes and thus to extract manageable information for human operators or subsequent systems figure 13–1 shows the interdependent steps of data analysis as described above.

The methods mentioned in the boxes in figure 13–1 indicate that numerous "classical" methods are already available. The process of data analysis described so far is not necessarily connected with fuzzy concepts.

If, however, either features or classes are fuzzy, the use of fuzzy approaches is desirable. In figure 13–1, for example, objects, features, and classes are considered. Both features and classes can be represented in crisp or fuzzy terms. An object is said to be fuzzy if at least one of its features is fuzzy. This leads to the following four cases:

- crisp objects and crisp classes
- crisp objects and fuzzy classes
- fuzzy objects and crisp classes
- fuzzy objects and fuzzy classes

Obviously, the first case is the domain of classical pattern recognition, while the latter three cases are the subject of fuzzy data analysis.

13.2 Methods for Fuzzy Data Analysis

Figure 13–1 indicates that some boxes—particularly those of feature analysis and classifier design—contain quite a number of classical dichotomous methods, such as clustering, regression analysis, etc., which for fuzzy data analysis have been fuzzified, i.e., modified to suit problem structures with fuzzy elements. The box

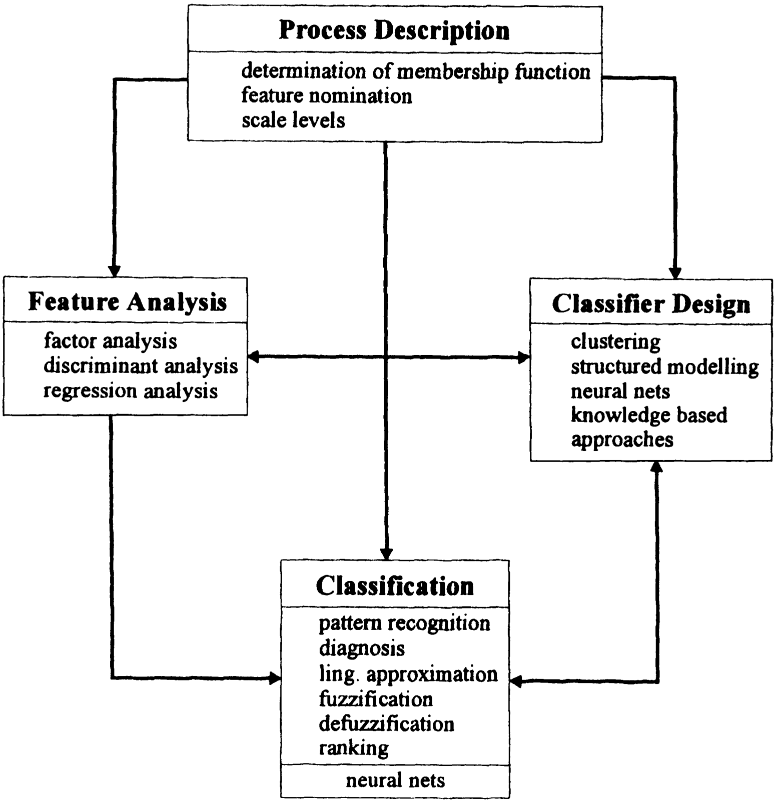


Figure 13–1. Scope of data analysis.

"classification," in contrast, lists some approaches that originate in fuzzy set theory and that did not exist before.

In modern fuzzy data analysis, three types of approaches can be distinguished. The first class is algorithmic approaches, which in general are fuzzified versions of classical methods, such as fuzzy clustering, fuzzy regression, etc. The second class is knowledge-based approaches, which are similar to fuzzy control or fuzzy expert systems. The third class, (fuzzy) neural net approaches, is growing rapidly in number and power. Increasingly combined with these approaches, but not discussed in this book, are evolutionary algorithms and genetic algorithms (see Zimmermann [1994]).

The major three classes mentioned above will be discussed in the following sections of this chapter.

13.2.1 Algorithmic Approaches

For feature analysis, fuzzy regression methods have been used. Recommended publications concerning this approach (which will not be discussed in this book) are, e.g., Bardossey et al. [1992, 1993], Diamond [1993], Ishibuchi [1992], Kacprzyk [1992], Peters [1994], and Tanaka [1987].

Here we shall focus our attention on clustering methods.

13.2.1.1 Fuzzy Clustering

13.2.1.1.1 Clustering Methods. Let us assume that the important problem of feature extraction—that is, the determination of the characteristics of the physical process, the image of other phenomena that are significant indicators of structural organization, and how to obtain these—has been solved. Our task is then to divide n objects $x \in X$ characterized by p indicators into c, $2 \leq c < n$, categorically homogenous subsets called "clusters." The objects belonging to any one of the clusters should be similar and the objects of different clusters as dissimilar as possible. The number of clusters, c, is normally not known in advance.

The most important question to be answered before applying any clustering procedure is which mathematical properties of the data set (for example, distance, connectivity, intensity, and so on) should be used and in what way they should be used in order to identify clusters. This question will have to be answered for each specific data set, since there are no universally optimal cluster criteria. Figure 13–2 shows a few possible shapes of clusters; and it should be immediately obvious that a cluster criterion that works in figure 13–2a will show a very bad performance in figures 13–2b or 13–2c. More examples can, for instance, be found in Bezdek [1981] or Roubens [1978] and in many other publications on cluster analysis and pattern recognition [Ismail 1988, p. 446; Gu and Dubuisson 1990, p. 213].

For further illustration of this point, let us look at an example from Bezdek [1981, p. 45]. Figure 13–3 shows two data sets, which have been clustered by a distance-based objective function algorithm (the within-group sum-of-spared-error criterion) and by applying a distance-based graph-theoretic method (single-linkage algorithm). Obviously, the criterion that leads to good results in one case performs very badly in the other case and vice versa. (Crisp) clustering methods are commonly categorized according to the type of clustering criterion used in hierarchical, graph-theoretic, and objective-functional methods.

Hierarchical clustering methods generate a hierarchy of partitions by means of a successive merging (agglomerative) or splitting (diverse) of clusters [Dimitrescu 1988, p. 145]. Such a hierarchy can easily be represented by a dendogram, which might be used to estimate an appropriate number of clusters, c,

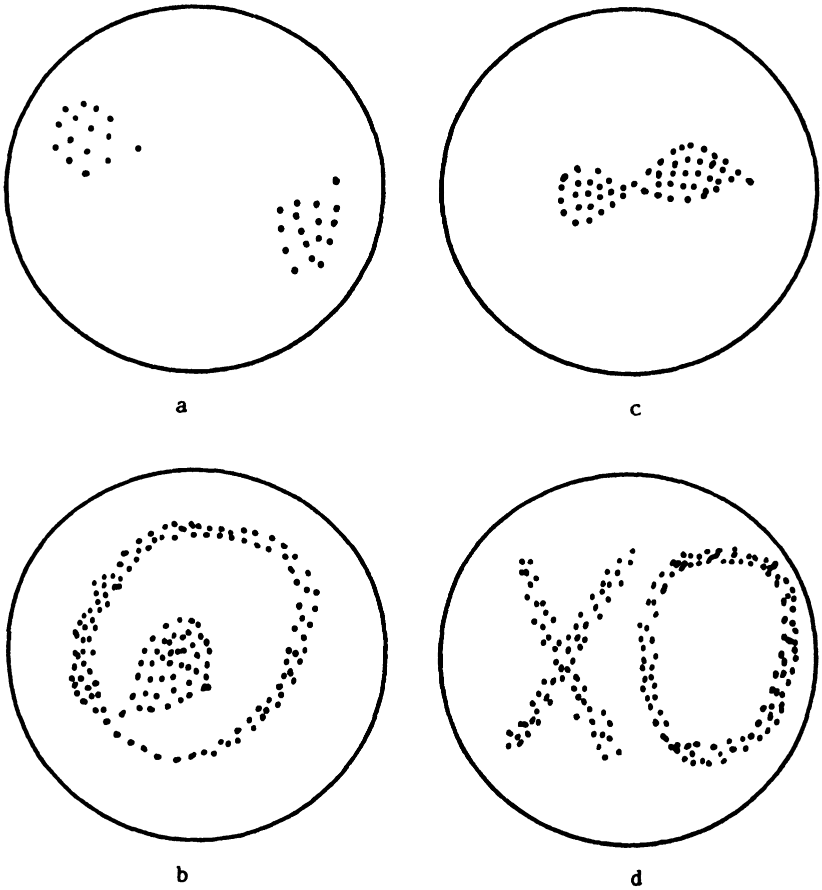


Figure 13–2. Possible data structures in the plane.

for other clustering methods. On each level of agglomeration or splitting, a locally optimal strategy can be used without taking into consideration the policies used on preceding levels. These methods are not iterative; they cannot change the assignment of objects to clusters made on preceding levels. Figure 13–4 shows a dendogram that could be the result of a hierarchical clustering algorithm. The main advantage of these methods is their conceptual and computational simplicity.

In fuzzy set theory, this type of clustering method would correspond to the determination of "similarity trees" such as those shown in example 6–14.

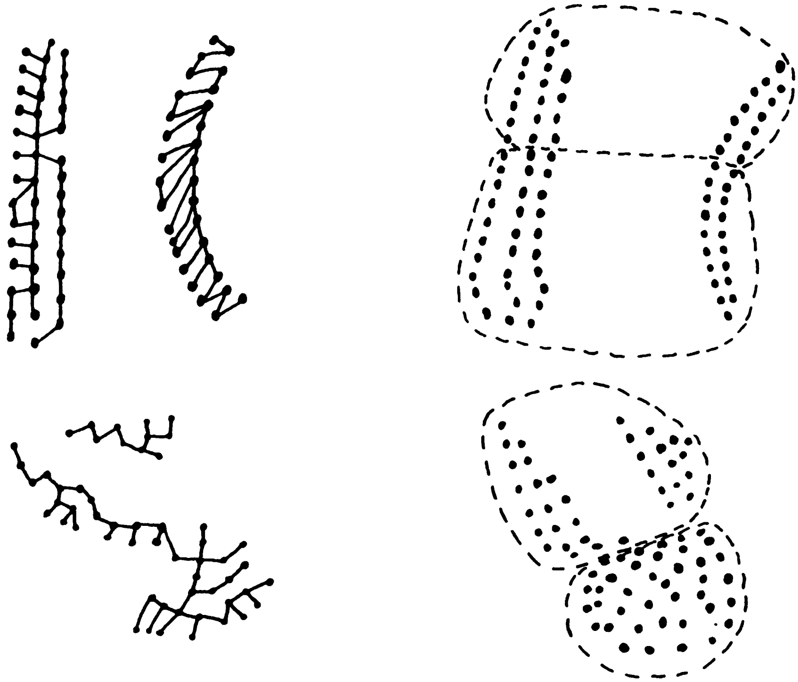


Figure 13–3. Performance of cluster criteria.

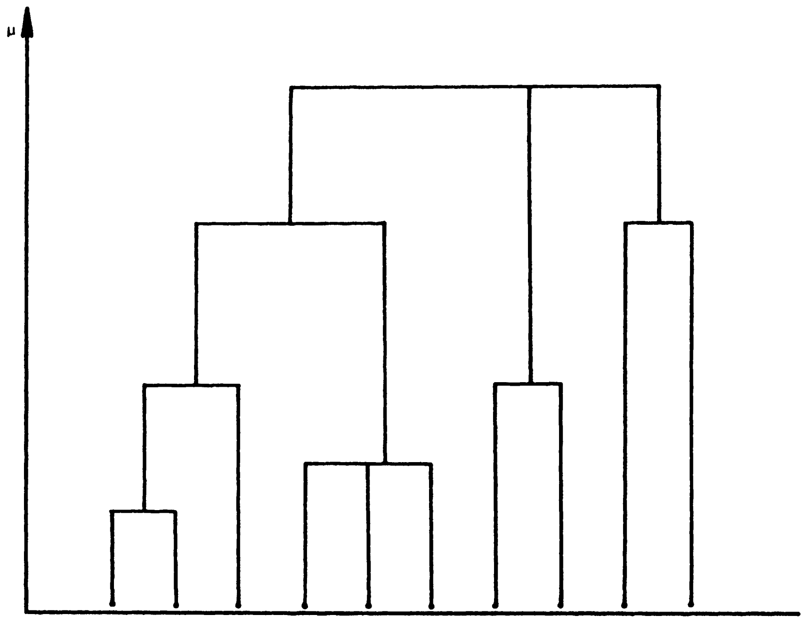


Figure 13–4. Dendogram for hierarchical clusters.

Graph-theoretic clustering methods are normally based on some kind of connectivity of the nodes of a graph representing the data set. The clustering strategy is often breaking edges in a minimal spanning tree to form subgraphs. If the graph representing the data structure is a fuzzy graph such as those discussed in chapter 6, then different notions of connectivity lead to different types of clusters, which in turn can be represented as dendograms. Yeh and Bang [1975], for instance, define four different kinds of clusters. For the purpose of illustrating this approach, we shall consider one of the types of clusters suggested there.

Definition 13–1 [Yeh and Bang 1975]

Let $\tilde{G} = [V, \tilde{R}]$ be a symmetric fuzzy graph. Then the *degree of a vertex* v is defined as $d(v) = \Sigma_{u \neq \omega} \mu_{\tilde{R}}(u)$. The minimum degree of $\tilde{G}$ is $\delta(\tilde{G}) = \min_{v \in V} \{d(v)\}$.

Let $\tilde{G} = [V, \tilde{R}]$ be a symmetric fuzzy graph. $\tilde{G}$ is said to be *connected* if, for each pair of vertices u and v in V, $\mu_{\tilde{R}}(u, v) > 0$. $\tilde{G}$ is called *τ-degree connected* for some $\tau \geq 0$ if $\delta(\tilde{G}) \geq \tau$ and $\tilde{G}$ is connected.

Definition 13–2

Let $\tilde{G} = [V, \tilde{R}]$ be a symmetric fuzzy graph. *Clusters* are then defined as maximal τ-degree connected subgraphs of $\tilde{G}$.

Example 13–1 [Yeh and Bang 1975, p. 145]

Let $\tilde{G}$ be the symmetric fuzzy graph shown in figure 13–5. The dendogram in figure 13–6 shows all clusters for different levels of τ. For further details, see Yeh and Bang [1975].

Objective-function methods allow the most precise formulation of the clustering criterion. The "desirability" of clustering candidates is measured for each c, the number of clusters, by an objective function. Typically, local extrema of the objective function are defined as optimal clusterings. Many different objective functions have been suggested for clustering (crisp clustering as well as fuzzy clustering). The interested reader is referred in particular to the excellent book by Bezdek [1981] for more details and many references. We shall limit our considerations to one frequently used type of (fuzzy) clustering method, the so-called c-means algorithm.

Classical (crisp) clustering algorithms generate partitions such that each object is assigned to exactly one cluster. Often, however, objects cannot adequately be assigned to strictly one cluster (because they are located "between" clusters). In

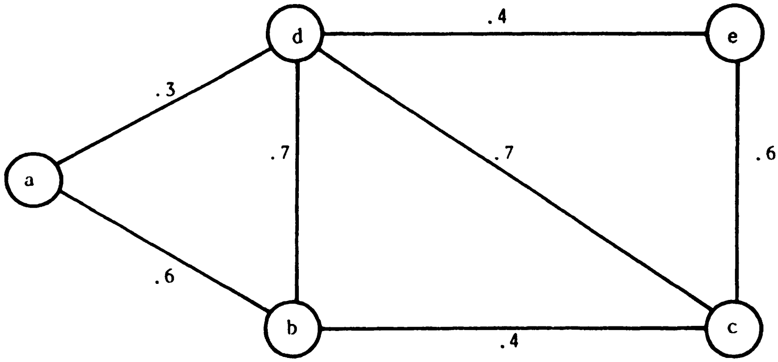


Figure 13–5. Fuzzy graph.

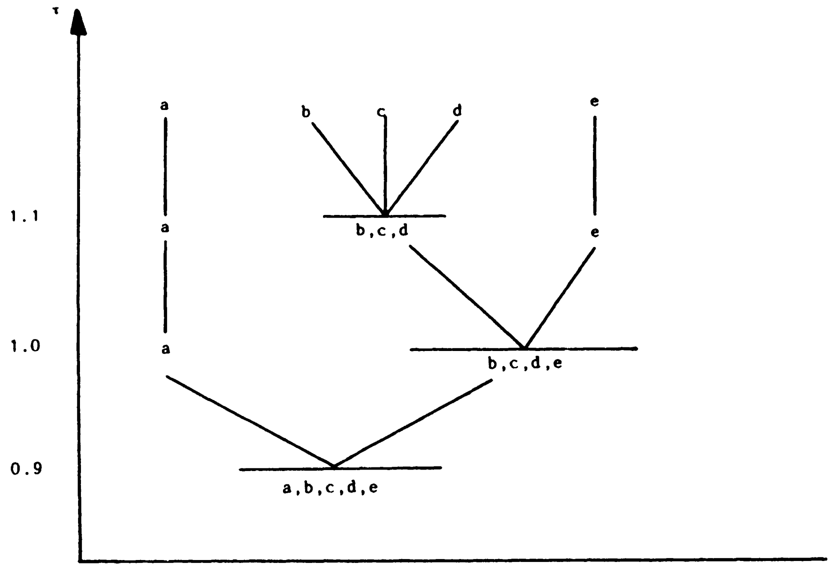


Figure 13–6. Dendogram for graph-theoretic clusters.

these cases, fuzzy clustering methods provide a much more adequate tool for representing real-data structures.

To illustrate the difference between the results of crisp and fuzzy clustering methods let us look at one example used in the clustering literature very extensively: the butterfly.

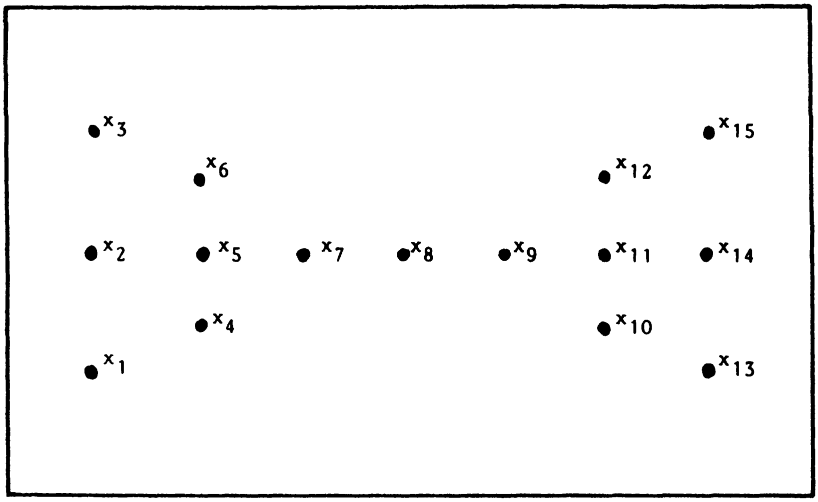


Figure 13–7. The butterfly.

Example 13–2

The data set X consists of 15 points in the plane, as depicted in figure 13–7. Clustering these points by a crisp objective-function algorithm might yield the picture shown in figure 13–8, in which "1" indicates membership of the point in the left-hand cluster and "0" membership in the right-hand cluster. The x's indicate the centers of the clusters. Figures 13–9 and 13–10, respectively, show the degrees of membership the points might have to the two clusters when using a fuzzy clustering algorithm.

We observe that, even though the butterfly is symmetric, the clusters in figure 13–8 are not symmetric because point x_8, the point "between" the clusters, has to be (fully) assigned to either cluster 1 or cluster 2. In figures 13–9 and 13–10, this point has the degree of membership .5 in both clusters, which seems to be more appropriate. Details of the methods used to arrive at figures 13–8 to 13–10 can be found in Bezdek [1981, p. 52] or Ruspini [1973].

Let us now consider the clustering methods themselves.

Let the data set $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^p$ be a subset of the real p-dimensional vector space $\mathbb{R}^p$. Each $x_k = (x_{k_1}, \ldots, x_{k_p}) \in \mathbb{R}^p$ is called a feature vector. x_{k_j} is the jth feature of observation x_k.

Since the elements of a cluster shall be as similar to each other as possible and the clusters as dissimilar as possible, the clustering process is controlled by use

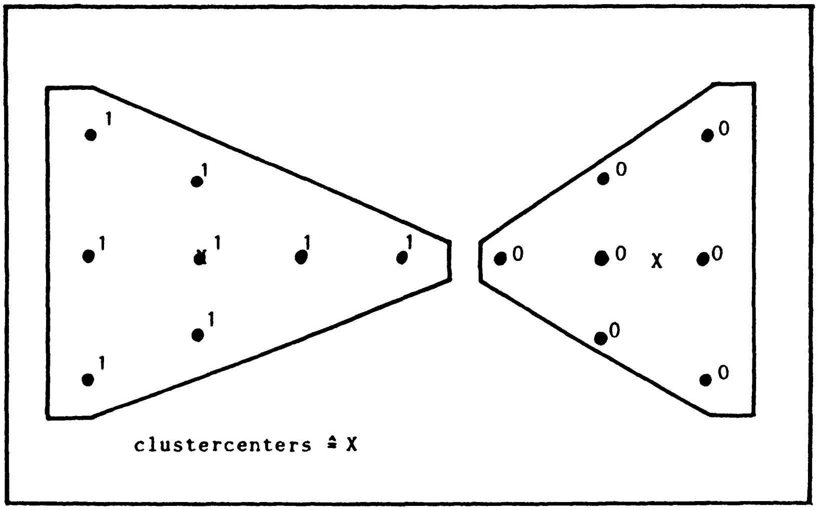


Figure 13–8. Crisp clusters of the butterfly.

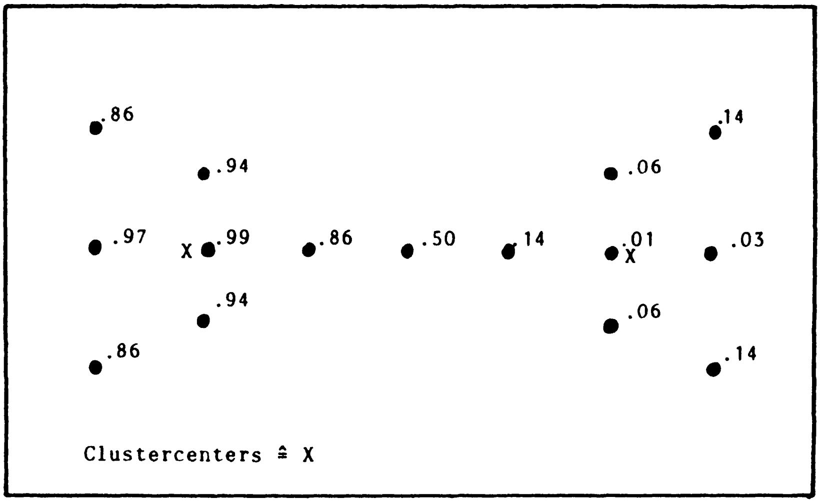


Figure 13–9. Cluster 1 of the butterfly.

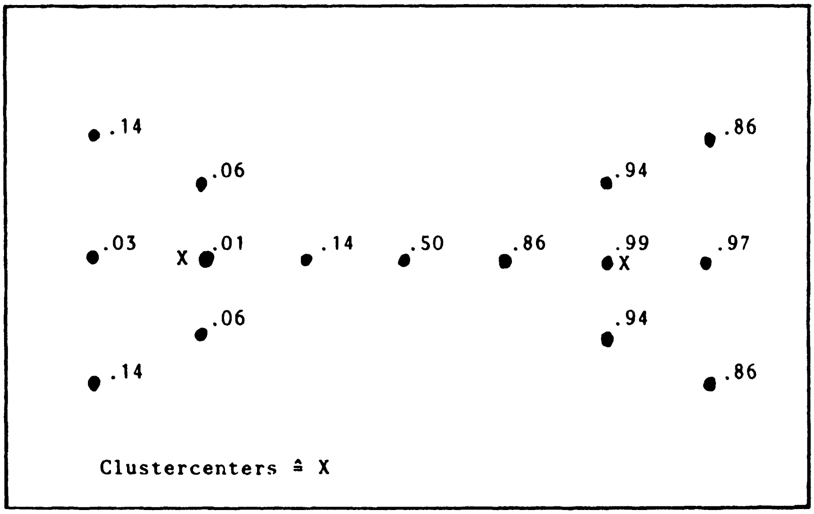


Figure 13–10. Cluster 2 of the butterfly.

of similarity measures. One normally defines the "dissimilarity" or "distance" of two objects x_k and x_l as a real-valued function d: $X \times X \to R^+$ that satisfies

$$\begin{aligned} d(x_k, x_l) &= d_{kl} \geq 0 \\ d_{kl} &= 0 \Leftrightarrow x_k = x_l \\ d_{kl} &= d_{lk} \end{aligned}$$

If additionally d satisfies the triangle equality, that is,

$$d_{kl} \leq d_{kj} + d_{jl}$$

then d is a metric, a property that is not always required. If each feature vector is considered as a point in the p-dimensional space, then the dissimilarity d_{kl} of two points x_k and x_l can be interpreted as the distance between these points.

Each partition of the set $X = \{x_1, \ldots, x_n\}$ into crisp or fuzzy subsets $\tilde{S}_i$ $(i = 1, \ldots, c)$ can fully be described by an indicator function u_{S_i} or a membership function $\mu_{\tilde{S}_i}$, respectively. In order to stay in line with the terminology of the preceding chapters, we shall use, for crisp clustering methods,

$$u_{S_i}: X \to \{0, 1\}$$

and, for fuzzy cases,

$$\mu_{\tilde{S}_i}: X \to \{0, 1\}$$

where u_{ik} and μ_{ik} denote the degree of membership of object x_k in the subset $\tilde{S}_i$, that is,

$$u_{ik} := u_{S_i}(x_k)$$

$$\mu_{ik} := \mu_{\tilde{S}_i}(x_k)$$

Definition 13–3

Let $X = \{x_1, \ldots, x_n\}$ be any finite set. V_{cn} is the set of all real $c \times n$ matrices, and $2 \leq c < n$ is an integer. The matrix $U = [u_{ik}] \in V_{cn}$ is called a *crisp c-partition* if it satisfies the following conditions:

1. $u_{ik} \in \{0, 1\} \quad 1 \leq i \leq c, 1 \leq k \leq n$
2. $\sum_{i=1}^{c} \mu_{ik} = 1 \quad 1 \leq k \leq r$
3. $0 < \sum_{k=1}^{n} \quad < n \quad 1 \leq i \leq c$

The set of all matrices that satisfy these conditions is called M_c.

Example 13–3

Let $X = \{x_1, x_2, x_3\}$. Then there are the following three *crisp 2-partitions*:

$$U_1 = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$U_2 = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$U_3 = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Obviously, conditions (2) and (3) of the definition rule out the following partitions:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Definition 13–4

X, V_{cn}, and c are as in definition 13.3. $\tilde{U} = [\mu_{ik}] \in V_{cn}$ is called a *fuzzy-c partition* if it satisfies the following conditions [Bezdek 1981, p. 26]:

1. $\mu_{ik} \in [0, 1] \quad 1 \le i \le c,\ 1 \le k \le n$
2. $\sum_{i=1}^{c} \mu_{ik} = 1 \quad 1 \le k \le r$
3. $0 < \sum_{k=1}^{n} \mu_{ik} < n \quad 1 \le i \le c$

M_{fc} will denote the set of all matrices satisfying the above conditions. By contrast to the crisp c-partition, elements can now belong to several clusters and to different degrees. Conditions (2) and (3) just require that the "total membership" of an element is normalized to 1 and that the element cannot belong to more clusters than exist.

Example 13–4

Let $X = \{x_1, x_2, x_3\}$. Then there exist infinitely many possible fuzzy 2-partitions, such as

$$\tilde{u}_1 = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & .5 & 0 \\ 0 & .5 & 1 \end{bmatrix} \end{array}$$

$$\tilde{u}_2 = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} .8 & .5 & .2 \\ .2 & .5 & .8 \end{bmatrix} \end{array}$$

$$\tilde{u}_3 = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} .8 & 1 & .9 \\ .2 & 0 & .1 \end{bmatrix} \end{array}$$

and so on.

Our butterfly example (figure 13–7), for instance, could have the following partition:

$$\tilde{U} = \left\{ \begin{array}{ccccccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ .86 & .97 & .86 & .94 & .99 & .94 & .86 & .5 & .14 & .06 & .01 & .06 & .14 & .03 & .14 \\ .14 & .03 & .14 & .06 & .01 & .06 & .14 & .5 & .86 & .94 & .99 & .94 & .86 & .97 & .86 \end{array} \right\}$$

The location of a cluster is represented by its "cluster center" $v_i = (v_{il}, \ldots, v_{ip}) \in \mathbb{R}^p$, $i = 1, \ldots, c$, around which its objects are concentrated.

Let $v = (v_1, \ldots, v_c) \in \mathbb{R}^{cp}$ be the vector of all cluster centers, where the v_i in general do not correspond to elements of X.

One of the frequently used criteria to improve an initial partition is the so-called *variance criterion.* This criterion measures the dissimilarity between the points in a cluster and its cluster center by the Euclidean distance. This distance, d_{ik}, is then [Bezdek 1981, p. 54].

$$\begin{aligned} d_{ik} &= d(x_k, v_i) \\ &= \|x_k - v_i\| \\ &= \left[\sum_{j=1}^{p} (x_{kj} - v_{ij})^2\right]^{1/2} \end{aligned}$$

The variance criterion for crisp partitions corresponds to minimizing the sum of the variances of all variables j in each cluster i, with $|S_i| = n$, and yields

$$\min \sum_{i=1}^{c} \sum_{j=1}^{p} \frac{1}{|S_i|} \sum_{x_k \in S_i} (x_{kj} - v_{ij})^2 \Leftrightarrow$$

$$\min \frac{1}{n} \sum_{i=1}^{c} \sum_{x_k \in S_i} \sum_{j=1}^{p} (x_{kj} - v_{ij})^2$$

As indicated by the above transformation, the variance criterion corresponds—except for the factor $1/n$—to minimizing the sum of the squared Euclidean distances. The criterion itself amounts to solving the following problem:

$$\min z(S_1, \ldots, S_c; v) = \sum_{i=1}^{c} \sum_{x_k \in S_i} \|x_k - v_i\|^2$$

such that

$$v_i = \frac{1}{|S_i|} \sum_{x_k \in S_i} x_k$$

Using definition 13–3, the variance criterion for crisp c-partitions can be written as

$$\min z(\tilde{U}, v) = \sum_{i=1}^{c} \sum_{k=1}^{n} u_{ik} \|x_k - v_i\|^2$$

such that

$$\text{v}_i = \frac{1}{\sum_{k=1}^{n} u_{ik}} \sum_{k=1}^{n} (u_{ik}) x_k$$

For fuzzy c-partitions according to definition 13–4, the variance criterion amounts to solving the following problem:

$$\min z(U, \text{v}) = \sum_{i=1}^{c} \sum_{k=1}^{n} (\mu_{ik})^m \|x_k - v_i\|^2$$

such that

$$\text{v}_i = \frac{1}{\sum_{k=1}^{n} u_{ik}} \sum_{k=1}^{n} (u_{ik})^m x_k, \quad m > 1$$

Here v_i is the mean of the x_k m-weighted by their degrees of membership. That means that the x_k with high degrees of membership have a higher influence on v_i than those with low degrees of membership. This tendency is strengthened by m, the importance of which we will discuss in more detail at a later time. It was shown (see, for instance, Bock [1979a, p. 144]) that, given a partition $\tilde{U}$, v_i is best represented by the clusters $\tilde{S}_i$ as described above.

If we generalize the criterion concerning the used norm, the crisp clustering problem can be stated as follows: Let G be a $(p \times p)$ matrix, which is symmetric and positive-definite. Then we can define a general norm

$$\|x_k - \text{v}_i\|_G^2 = (x_k - \text{v}_i)^T G(x_k - \text{v}_i)$$

The possible influence of the chosen norm, determined by the choice of G, will be discussed later. This yields the formulation of the problem:

$$\min z(U, \text{v}) = \sum_{k=1}^{n} \sum_{i=1}^{c} u_{ik} \|x_k - \text{v}_i\|_G^2$$

such that

$$U \in M_c$$

$$\text{v} \in R^{cp}$$

This is a combinatorial optimization problem that is hard to solve, even for rather small values of c and n. In fact, the number of distinct ways to partition x into nonempty subsets is

$$|M_c| = (1/c!)\left[\sum_{j=1}^{c} \binom{c}{j}(-1)^{c-j} j^n\right]$$

which for $c = 10$ and $n = 25$ is already roughly 10^{18} distinct 10-partitions of the 25 points [Bezdek 1981, p. 29].

The basic definition of the fuzzy partitioning problem for $m > 1$ is

$$\min z_m(\tilde{U}; \mathrm{v}) = \sum_{k=1}^{n}\sum_{i=1}^{c} (\mu_{ik})^m \|x_k - \mathrm{v}_i\|_G^2 \qquad (P_m)$$

such that

$$\tilde{U} \in M_{fc}$$

$$\mathrm{v} \in R^{cp}$$

(P_m) is an analytical problem, which has the advantage that by using differential calculus one can determine necessary conditions for local optima. Differentiating the objective function with respect to v_i (for fixed $\tilde{U}$) and to μ_{ik} (for fixed v) and applying the condition $\sum_{i=1}^{c}\mu_{ik} = 1$, one obtains (see [Bezdek 1981, p. 67]):

$$\mathrm{v}_i = \frac{1}{\sum_{k=1}^{n}(\mu_{ik})^m} \sum_{k=1}^{n} (\mu_{ik})^m x_k \quad i = 1, \ldots, c \qquad (13.1)$$

$$\mu_{ik} = \frac{\left(\dfrac{1}{\|x_k - \mathrm{v}_i\|_G^2}\right)^{1/(m-1)}}{\sum_{j=1}^{c}\left(\dfrac{1}{\|x_k - \mathrm{v}_j\|_G^2}\right)^{1/(m-1)}}, i = 1, \ldots, c; k = 1, \ldots, n \qquad (13.2)$$

Let us now comment on the role and importance of m: It is called the exponential weight, and it reduces the influence of "noise" when computing the cluster centers in equation (13.1) (see Windham [1982, p. 358]) and the value of the objective function z_m $(U; \mathrm{v})$. m reduces the influence of small μ_{ik} (points further away from v_i) compared to that of large μ_{ik} (points close to v_i). The larger $m >$ 1, the stronger is this influence.

The systems described by equations (13.1) and (13.2) cannot be solved analytically. There exist, however, iterative algorithms (nonhierarchical) that approximate the minimum of the objective function, starting from a given position. One of the best-known algorithms for the crisp clustering problem is the (hard) c-means algorithm or (basic) ISODATA-algorithm. Similarly, the fuzzy clustering problem can be solved by using the fuzzy c-means algorithm, which shall be described in more detail in the following.

The *fuzzy c-means algorithm* [Bezdek 1981, p. 69]. For each $m \in (0, \infty)$, a fuzzy c-means algorithm can be designed that iteratively solves the necessary conditions (13.1) and (13.2) above and converges to a local optimum (for proofs of convergence, see Bezdek [1981] and Bock [1979]).

The algorithm comprises the following steps:

Step 1. Choose c $(2 \le c \le n)$, m $(1 < m < \infty)$, and the (p, p)-matrix G with G symmetric and positive-definite. Initialize $\tilde{U}^{(0)} \in M_{fc}$, set $l = 0$.

Step 2. Calculate the c fuzzy cluster centers $\{v_i^{(l)}\}$ by using $\tilde{U}^{(l)}$ from condition (13.1).

Step 3. Calculate the new membership matrix $\tilde{U}^{(l+1)}$ by using $\{v_i^{(l)}\}$ from condition (13.2) if $x_k \neq v_i^{(l)}$. Else set

$$\mu_{jk} = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

Step 4. Choose a suitable matrix norm and calculate $\Delta = \| \tilde{U}^{(l+1)} - \tilde{U}^{(l)} \|_G$. If $\Delta > \varepsilon$, set $l = l + 1$ and go to step 2. If $\Delta \le \varepsilon$, $\rightarrow$ stop.

For the fuzzy c-means algorithm, a number of parameters have to be chosen:

the number of clusters c, $2 \le c \le n$;
the exponential weight m, $1 < m < \infty$;
the (p, p) matrix G (G symmetric and positive-definite), which induces a norm;
the method to initialize the membership matrix $\tilde{U}^{(0)}$;
the termination criteria $\Delta = \| \tilde{U}^{(l+1)} - \tilde{U}^{(l)} \|_G \le \varepsilon$.

Example 13–5 [Bezdek 1981, p. 74]

The data of the butterfly shown in figure 13–7 were processed with a fuzzy 2-means algorithm, using as a starting partition

$$\tilde{U}^{(0)} = \begin{bmatrix} .854 & .146 & .854 & \dots & .854 \\ .146 & .854 & .146 & \dots & .146 \end{bmatrix}_{2\times 15}$$

ε was chosen to be .01; the Euclidean norm was used for G; and m was set to 1.25. Termination in six iterations resulted in the memberships and cluster centers shown in figure 13–11. For $m = 2$, the resulting clusters are shown in figure 13–12.

As for other iterative algorithms for improving starting partitions, the number c has to be chosen suitably. If there does not exist any information about a good c, the computations are carried out for several values of c. In a second step, the best of these partitions is selected.

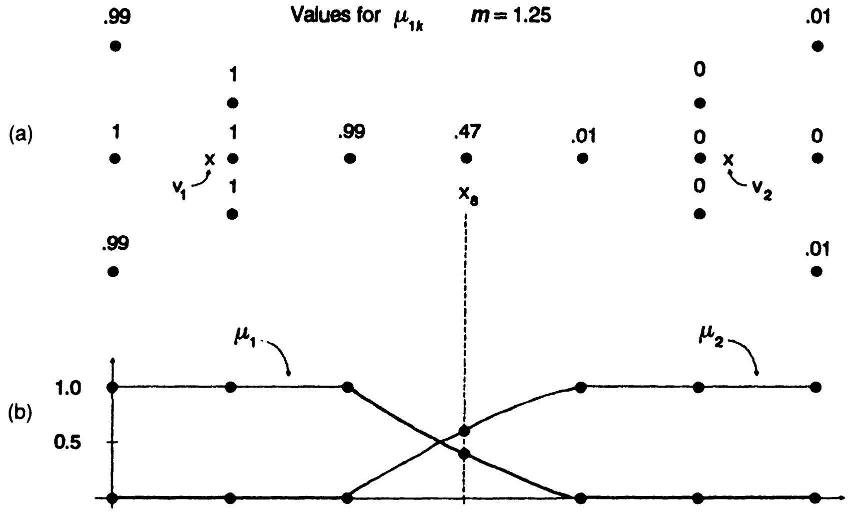


Figure 13–11. Clusters for $m = 1.25$.

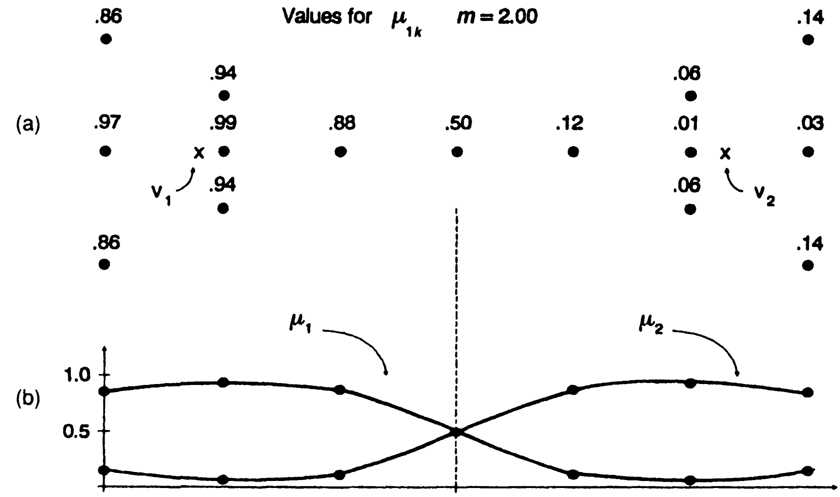


Figure 13–12. Clusters for $m = 2$.

The exponential weight m influences the membership matrix. The larger the m, the fuzzier becomes the membership matrix of the final partition. For $m \to \infty$, $\tilde{U}$ approaches $\tilde{U} = [\frac{1}{c}]$. This is, of course, a very undesirable solution, because each x_k is assigned to each cluster with the same degree of membership.

Basically, less fuzzy membership matrices are preferable because higher degrees of membership indicate a higher concentration of the points around the respective cluster centers. No theoretically justified rule for choosing m exists. Usually $m = 2$ is chosen.

G determines the shape of the cluster, which can be identified by the fuzzy c-means algorithm. If one chooses the Euclidean norm N_E, then G is the identity matrix I, and the shape of the clusters is assumed to be an equally sized hypersphere. Other frequently used norms are the diagonal norm or the Mahalanobis norm for which $G_D = [\mathrm{diag}(\sigma_j^2)]^{-1}$ and $G_M = [\mathrm{cov}\ (x)]^{-1}$, respectively, where σ_j^2 denotes the variance of feature j.

The final partition depends on the initially chosen starting position. When choosing an appropriate c, if there exists a good clustering structure, the final partitions generated by a fuzzy c-means algorithm are rather stable.

A number of variations of the above algorithm are described in Bezdek [1981]. The interested reader is referred to this reference for further details. Numerical results for a number of algorithms are also presented in Roubens [1978].

13.2.1.1.2 Cluster Validity

> Complex algorithms stand squarely between the data for which substructure is hypothesized and the solutions they generate; hence it is all but impossible to transfer a theoretical null hypothesis about X to $\tilde{U} \in M_{fc}$, which can be used to statistically substantiate or repudiate the validity of algorithmically suggested clusters. As a result a number of scalar measures of partition fuzziness (which are interesting in their own right) have been used as heuristic validity indicants [Bezdek 1981, p. 95].

Actually, the so-called cluster validity problem concerns the quality or the degree to which the final partition of a cluster algorithm approximates the real or hypothesized structure of a set of data. Most often this question is reduced, however, to the search for a "correct" c. Cluster validity is also relevant when deciding which of a number of starting partitions should be selected for improvement.

For measuring cluster validity in fuzzy clustering, some criteria from crisp cluster analysis have been adapted to fuzzy clustering. In particular, the so-called validity functionals used express the quality of a solution by measuring its degree of fuzziness. While criteria for cluster validity are closely related to the mathematical formulation of the problem, criteria to judge the real "appropriateness" of a final partition consider primarily real rather than mathematical features.

Let us first consider some criteria taken from traditional crisp clustering.

One of the most straightforward criteria is the value of the objective function. Since it decreases monotonically with increasing number of clusters, c, that is, it reaches its minimum for $c = n$, one chooses the c^* for which a large decrease is obtained when going from c^* to $c^* + 1$. Another criterion is the rate of convergence. This is justified because experience has shown that, for a good clustering structure and for an appropriate c, a high rate of convergence can generally be obtained.

Because the "optimal" final portion depends on the initialization of the starting partition $\tilde{U}^0$, the "stability" of the final partition with respect to different starting partitions can also be used as an indication of a "correct" number of clusters c.

All three criteria serve to determine the "correct" number of clusters. They are heuristic in nature and therefore might lead to final partitions that do not correctly identify existing clusters. Bezdek shows, for instance, that the global minimum of the objective function is not necessarily reached for the correct partition [Bezdek 1981, pp. 96 ff]. Therefore other measures of cluster validity are needed in order to judge the quality of a partition.

The following criteria calculate cluster validity functionals that assign to each fuzzy final partition a scalar that is supposed to indicate the quality of the clustering solution. When designing such criteria, one assumes that the clustering structure is better identified when more points concentrate around the cluster centers, that is, the crisper (unfuzzier) is the membership matrix of the final partition generated by the fuzzy c-means algorithm.

The best-known measures for judging the fuzziness of a clustering solution are

the partition coefficient, $F(\tilde{U}, c)$,
the partition entropy, $H(\tilde{U}, c)$, and
the proportion exponent, $P(\tilde{U}, c)$.

Definition 13–5 [Bezdek 1981, p. 100]

Let $\tilde{U} \in M_{fc}$ be a fuzzy c-partition of n data points. The *partition coefficient* of $\tilde{U}$ is the scalar

$$F(\tilde{U}, c) = \sum_{k=1}^{n} \sum_{i=1}^{c} \frac{(\mu_{ik})^2}{n}$$

Definition 13–6 [Bezdek 1981, p. 111]

The *partition entropy* of any fuzzy c-partition $\tilde{U} \in M_{fc}$ of X, where $|X| = n$, is for $1 \le c \le n$

$$H(\tilde{U}, c) = -\frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{c}\mu_{ik}\log_e(\mu_{ik})$$

(see definition 4–3a, b, where the entropy was already used as a measure of fuzziness.)

Definition 13–7 [Windham 1981, p. 178; Bezdek 1981, p. 119]

Let $\tilde{U} \in (M_{fc} \backslash M_{c0})$ be a fuzzy c-partition of X; $|X| = n$; $2 \le c < n$. For column k of $\tilde{U}$, $1 \le k \le n$, let

$$\mu_k = \max_{1 \le i \le c}\{\mu_{ik}\}$$

$$[\mu_k^{-1}] = \text{greatest integer} \le \left(\frac{1}{\mu_k}\right)$$

The *proportion exponent* of U is the scalar

$$P(\tilde{U}, c) = -\log_e\left\{\prod_{k=1}^{n}\left[\sum_{j=1}^{\mu_k^{-1}}(-1)^{j+1}\binom{c}{j}(1 - j\mu_k)^{(c-1)}\right]\right\}$$

The above-mentioned measures have the following properties:

$$\frac{1}{c} \le F(\tilde{U}, c) \le 1$$

$$0 \le H(\tilde{U}, c) \le \log_e(c)$$

$$0 \le P(\tilde{U}, c) < \infty$$

The partition coefficient and the partition entropy are similar in so far as they attain their extrema for crisp partitions $U \in M_c$:

$$F(\tilde{U}, c) = 1 \Leftrightarrow H(\tilde{U}, c) = 0 \Leftrightarrow \tilde{U} \in M_c$$

$$F(\tilde{U}, c) = \frac{1}{c} \Leftrightarrow H(\tilde{U}, c) = \log_e(c) \Leftrightarrow \tilde{U} = \left[\frac{1}{c}\right]$$

The (heuristic) rules for selecting the "correct" or best partitions are

$$\max_c \{\max_{\tilde{U} \in \Omega_c}\{F(\tilde{U}, c)\}\} \quad c = 2, \ldots, n-1$$

$$\min_c \{\min_{\tilde{U} \in \Omega_c}\{H(\tilde{U}, c)\}\} \quad c = 2, \ldots, n-1$$

where Ω_c is the set of all "optimal" solutions for given c.

The limitations of $F(\tilde{U}, c)$ and $H(\tilde{U}, c)$ are mainly their monotonicity and the lack of any suitable benchmark that would allow a judgment as to the acceptability of a final partition. The monotonicity will usually tend to indicate that the "correct" partition is the 2-partition. This problem can be solved, for instance, by choosing the i^* partition for which the value of $H(\tilde{U}, c)$ lies below the trend when going from $c^* - 1$ to c^*.

$H(\tilde{U}, c)$ is normally more sensitive with respect to a change of the partition than is $F(\tilde{U}, c)$. This is particularly so if m is varied.

While $F(\tilde{U}, c)$ and $H(\tilde{U}, c)$ depend on all $c \cdot n$ elements, the proportion exponent $P(\tilde{U}, c)$ depends on the maximum degree of membership of the n elements. $P(\tilde{U}, c)$ converges towards ∞ with increasing μ_k, and it is not defined for $\mu_k = 1$.

The heuristic for choosing a good partition is

$$\max_{c} \{\max_{\tilde{U} \in \Omega_c}\{P(\tilde{U}, c)\}\} \quad c = 2, \ldots, n-1$$

By contrast to $F(\tilde{U}, c)$ and $H(\tilde{U}, c)$, $P(\tilde{U}, c)$ has the advantage that it is not monotone in c. There exist, however, no benchmarks such that one can judge the quality of a portion c^* from the value of $P(\tilde{U}, c^*)$.

The heuristic for $P(\tilde{U}, c)$ possibly leads to an "optimal" final partition other than the heuristics of $F(\tilde{U}, c)$ and/or of $H(\tilde{U}, c)$. This might necessitate the use of other decision aids derived from the data themselves or from other considerations. Bezdek [1981] describes quite a number of other approaches in his book.

Even though the fuzzy c-means algorithm (FCM) performs better in practice than crisp clustering methods, problems may still have features that cannot be a-ccommodated by the FCM. Exemplarily, two of them shall be looked at briefly.

Most crisp and fuzzy clustering algorithms seek in a set of data one or the other type of clustershape (prototype). The type of prototype used determines the distance measurement criteria used in the objective function. Windham [1983] presented a general procedure that unifies and allows the construction of different algorithms using points, lines, planes, etc. as prototypes. These algorithms, however, normally fail, if the pattern looked for is not in sense compact. For instance, the patterns shown figures 13–2b and 13–2c will hardly be found. Dave [1990] suggested an algorithm that can find rings or, in general, spherical shells in higher dimensions. His fuzzy shell clustering (FSC) algorithm modifies the variance criterion mentioned above (after example 13–4) by introducing the radius of the "ring" searched for, arriving at

$$\min z_s(u, v, r) = \sum_{i=1}^{c} \sum_{k=1}^{n} (\mu_{ik})^m (D_{ik})^2$$

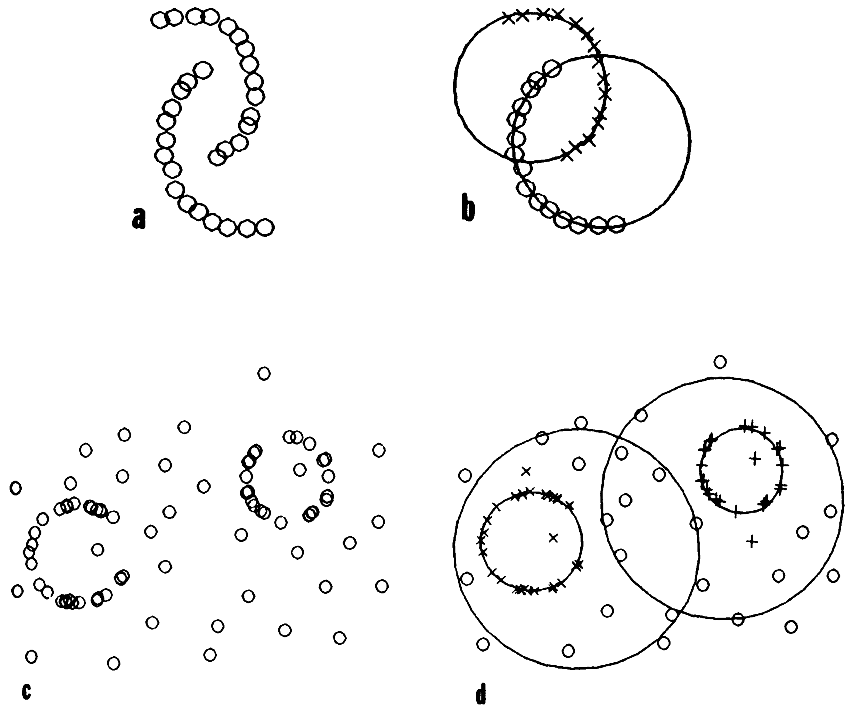


Figure 13–13. Clusters by the FSC. (a) Data set; (b) circles found by FSC; (c) data set; (d) circles found by FSC.

where

$$D_{ik} = \left| \|x_k - v_i\| - r_i \right|$$

r_i is the radius of the cluster prototype shell, and all other symbols are as defined for the FCM algorithm. The algorithm itself has to be adjusted accordingly by including r_i.

Details are given in Dave [1990]. This algorithm also finds circles if the data are incomplete. Figure 13–13 shows examples of it from Dave.

Interesting applications can be found in Dave and Fu [1994].

The FCM as well as the FSC satisfies the constraint

$$\sum_{i=1}^{c} \mu_{ik} = 1, \quad 1 \le k \le n$$

which was used in definition 13–4 of a fuzzy c-partition. Considering data sets shown in figure 13–14, this constraint would enforce that, for instance, two cluster

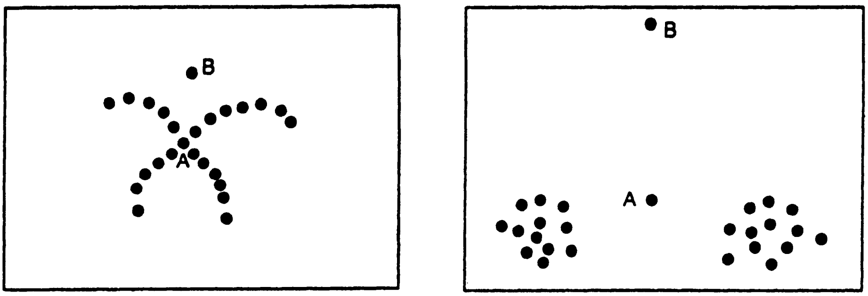


Figure 13–14. Data sets [Krishnapuram and Keller 1993].

points A and B would get the same degree of membership, $\mu = .5$, in clusters 1 and 2.

The μ_{ik} would then express a kind of "relative membership" to the clusters, i.e., the membership of point B in cluster 1 compared to the membership of point B in cluster 2 (see also figure 13–14). From an observer's point of view it might, however, be inappropriate to assign the same degrees of membership to points A and B because he interprets those as (absolute) degrees of membership, e.g., degrees to which points A or B belong to clusters 1 or 2, respectively. Krishnapuram and Keller [1993] suggest their possibilistic c-means algorithm (PCM) to compute the latter kind of degrees of membership for elements in clusters by modifying the definition of a fuzzy c-partition and, as a consequence, the objective function of the cluster algorithm.

Definition 13–4 is modified to

1. $\mu_{ik} \in [0, 1], \quad 1 \le i \le c, \quad 1 \le k \le n$
2. $0 < \sum_{k=1}^{n} \mu_{ik} \le n, \quad 1 \le i < c$
3. $\max_i \mu_{ik} > 0 \quad \text{for all } k.$

Simply relaxing condition 2 in definition 13–4 in the FCM would produce the trivial solution, i.e., the objective function would drive all degrees of membership to 0. This result is certainly not meaningful. One would rather try to have the degrees of membership of data that belong strongly to clusters appropriately high and those that do not represent the features of the clusters well very low. This is achieved by the following objective function:

$$\min z(\tilde{U}, v) = \sum_{i=1}^{c} \sum_{k=1}^{n} (\mu_{ik})^m d_{ik}^2 + \sum_{i=1}^{c} \eta_i \sum_{k=1}^{n} (1 - \mu_{ik})^m.$$

Here d_{ik} can be the same distance as in the FCM, μ_{ik} are now the "absolute" degrees of memberships, and η_i are appropriately chosen positive numbers (see

Krishnapuram and Keller [1993]). When applying such an algorithm to data sets as shown in figure 13–14, point A would obtain considerably higher degrees of membership than point B.

13.2.2 Knowledge-Based Approaches

Knowledge-based approaches resemble very much those procedures described in chapters 10 and 11. Figure 13–15 indicated the basic structure of knowledge-based classification.

After the preprocessing, the data describing the elements are fed into an expert system. This contains in the knowledge base—in an appropriate fuzzy description—the relevant features, which in the inference engine are aggregated per element. The results are either membership functions or possibly singletons. The "matching" function contains the description of the classes (fuzzy or crisp) and determines the similarity of the expert system output with the class description. An assignment of elements to classes occurs then either according to the respective degrees of similarity or to the class with the highest degree of similarity.

An example of such a data-mining system is described by Fei and Jawahir [1992]. The basic structure is given below.

In a turning situation, the finish-turning operation involving the machining of a component at small feeds and at small depths of cut requires a number of major issues to be solved before the process can begin. The process of finish turning itself is so complex that it is practically impossible to establish any theoretical model that could precisely predict the machinability parameters. Here we shall only consider the relationship between depth of cut and feed on one hand and the resulting surface roughness on the other hand.

Figure 13–16 shows the linguistic variables defining the relevant features on the input side.

In this case the classes, i.e., surface roughness, are defined as intervals with linguistic labels as follows:

Label	*Excellent*	*Good*	*Fair*	*Acceptable*	*Poor*
R_a (μm)	.0–.6	.6–1.1	1.1–1.5	1.5–2.0	2.0–3.0

The authors have modeled the uncertainty in this case by computing a kind of "uncertainty factor" that applies to the respective terms of the linguistic variable (classes). Alternatively, the classes could, of course, have been modeled by fuzzy sets, rather than by intervals, possibly in multidimensional space.

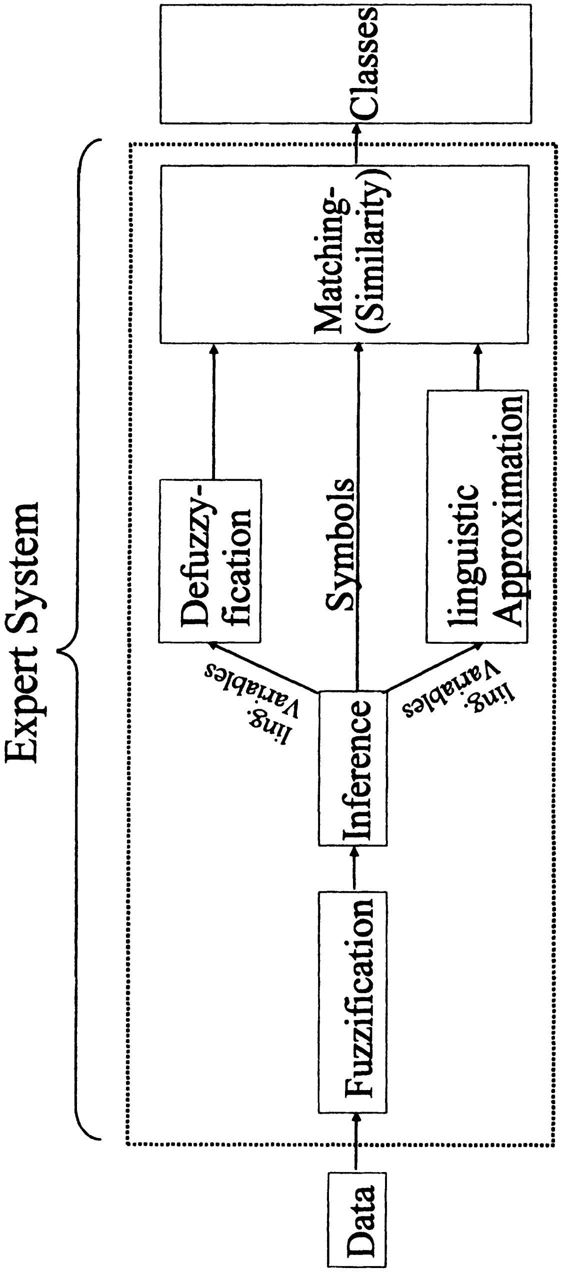


Figure 13–15. Knowledge-based classification.

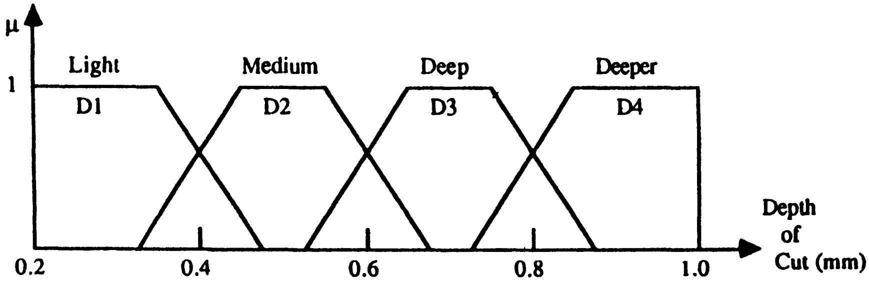


Figure 13–16. Linguistic variables "Depth of Cut" and "Feed."

Depth \ Feed	F1	F2	F3	F4	F5	F6	F7	F8
Deeper (D4)	1.0/E	1.0/G	0.6/G 0.4/F	1.0/A	1.0/A	1.0/A	1.0/A	1.0/A
Deep (D3)	1.0/E	0.2/E 0.8/G	1.0/G	0.1/G 0.9/F	0.7/F 0.3/A	1.0/A	0.6/A 0.4/P	1.0/P
Medium(D2)	1.0/E	0.1/E 0.9/G	1.0/G	0.4/G 0.6/F	0.6/F 0.4/A	1.0/A	0.2/A 0.8/P	1.0/P
Light (D1)	1.0/E	1.0/G	1.0/G	0.4/G 0.6/F	1.0/F	0.3/F 0.7/A	1.0/A	0.9/A 0.1/P

E — Excellent, G — Good, F — Fair, A — Acceptable, P — Poor

Work Material = AISI 1045 Chip Breaker = FCB4
Cutting Speed = 230 m/min Tool Insert = TNMG 160408

Figure 13–17. Knowledge base.

The knowledge base of this system is shown in figure 13–17 and the structure of the entire system in figure 13–18.

13.2.3 Neural Net Approaches

Artificial neural nets (ANNs) have proven to be a very efficient and powerful tool for pattern recognition. The literature on types of ANNs and their applications to

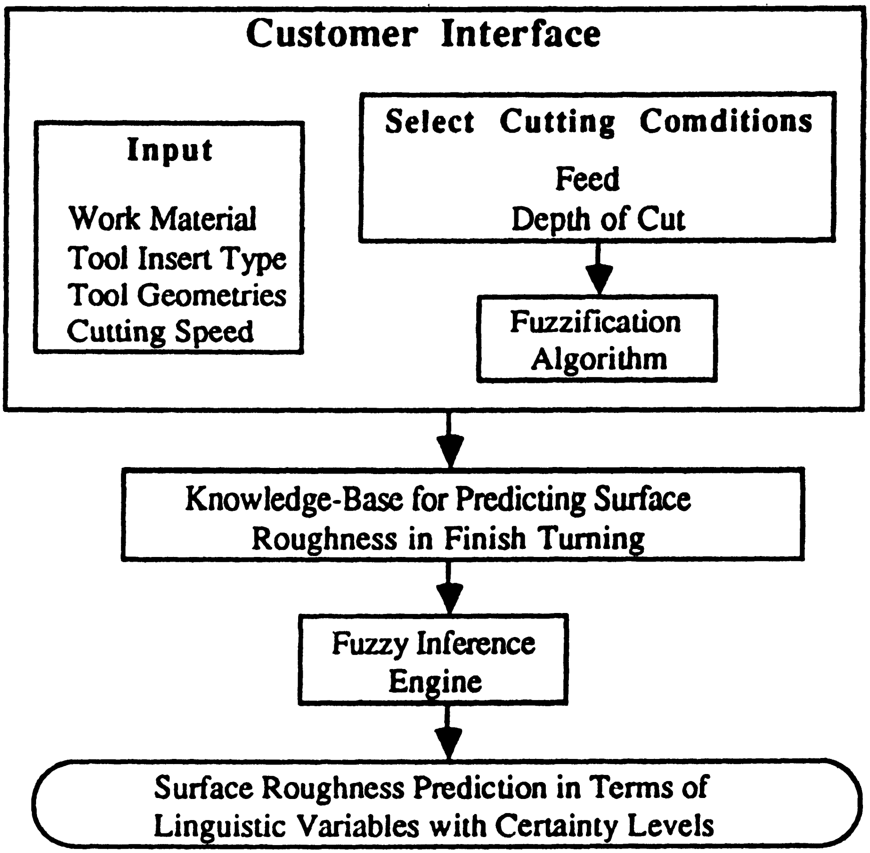


Figure 13–18. Basic structure of the knowledge-based system.

data analysis is abundant, and it would exceed the scope of this book to introduce the reader to this area. Since the beginning of the 1990s the relationship and the cross-fertilization of fuzzy set theory and artificial neural nets have grown stronger and stronger (see, for example, Lee [1975], Huntsberger [1990], Kosko [1992], Nauck et al. [1994], Kim and Choo [1994], and Kunchera [1994]). There are two reasons for this: (1) artificial neural nets are "classical" in the sense that originally their structure was dichotomous and a fuzzification has turned out to be useful in many cases, and (2) fuzzy set systems and ANNs are complementary in the sense that fuzzy systems are interpretable, plausible, and in a sense transparent (knowledge-based) systems, which, however, in general cannot learn. In other words, the knowledge has to be acquired first and then fed into the systems in the form of if-then rules or otherwise. ANNs, by contrast, have the "black box" character, i.e., they cannot be interpreted easily, but they can learn in a supervised or unsupervised fashion.

It is obvious that it makes sense to combine the attractive features of these two approaches while trying to avoid their weaknesses. Unfortunately, it is also beyond the scope of this book to describe the various ways in which these two approaches have been combined.

13.3 Dynamic Fuzzy Data Analysis

13.3.1 Problem Description

So far "objects" were considered to be elements or points (vectors) in the appropriate spaces.

The development of objects over time (and, therefore, the development of the features) is not considered explicitly or is taken into account by just using single values of the past in the feature vector.

Methods that use this type of feature vectors can be called static. In many applications, however, explicit consideration of trajectories rather than single points is desirable, e.g.:

- monitoring of patients in medicine, e.g. during narcosis, where the development of the patients' condition is essential;
- state-dependent machine maintenance;
- rating of shares: the examination of the development of share prices and other characteristics allows better estimates than just considering the current share price.

In all cases, where a dynamic viewpoint is desirable, the momentary snapshot for some components of the feature vector may be replaced by a trajectory of this feature. Thus, dynamic objects are represented by multi-dimensional trajectories in the feature space. Since most methods for data analysis are not suited to classify objects described by trajectories, new methods for dynamic data analysis were developed.

Figure 13–19 illustrates the difference between classical (static) and dynamic data analysis. Consider a two-dimensional feature space with one additional time dimension and suppose that a set of objects is observed over time. States of objects at a point of time can be seen in the cut of this three-dimensional space at the current moment (figure 13–19a). Two classes of objects can easily be distinguished in this plane. However, if the trace of each object from the initial to its current state (i.e. its trajectory) is considered and projected into the feature space (figure 13–19b), other classes may seem more reasonable.

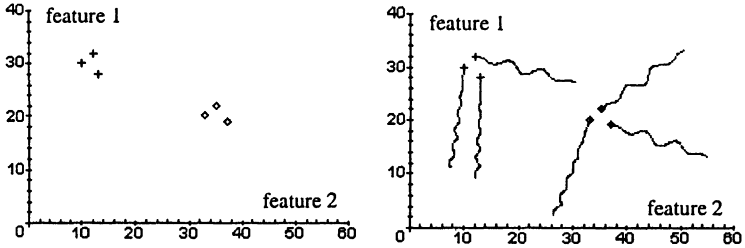


Figure 13–19. (a) States of objects at a point of time; (b) projections of trajectories over time into the feature space.

One of the major problems is that "distances" or "similarities" used in cluster algorithms are defined with respect to pairs of points, but not with respect to pairs of functions (or vectors).

13.3.2 Similarity of Functions

As stated before, the components of feature vectors describing dynamic objects are trajectories. Starting from the fact that most methods for data analysis use a distance measure or a similarity measure as a criterion to classify objects, one way to handle dynamic objects is to define the similarity measure for trajectories (functions) and to use it within existing or perhaps completely new methods.

Similarity of trajectories can be defined in different ways. Basically, two viewpoints can be distinguished.

The more similar are two trajectories

- the better they match in form/evolution/characteristics(*structural* similarity)
- the smaller their (pointwise) distance in feature space is (*pointwise* similarity).

Figure 13–20 gives an example of the differences between structural and pointwise similarity. In terms of pointwise similarity A and B would be grouped together as well as C and D. But in terms of structural similarity the grouping {A,D} and {B,C} seems to be more natural (depending on the chosen type of structural similarity). The following two sections describe these two types of similarity and the relationships between them.

Structural Similarity between Functions. Structural similarity relates to a variety of aspects of the trajectories (functions) under consideration: form, evolution, size or orientation (of trajectories in $\mathcal{R}^n$)are some examples. Depending on the chosen aspect, different criteria may be relevant to describe similar-

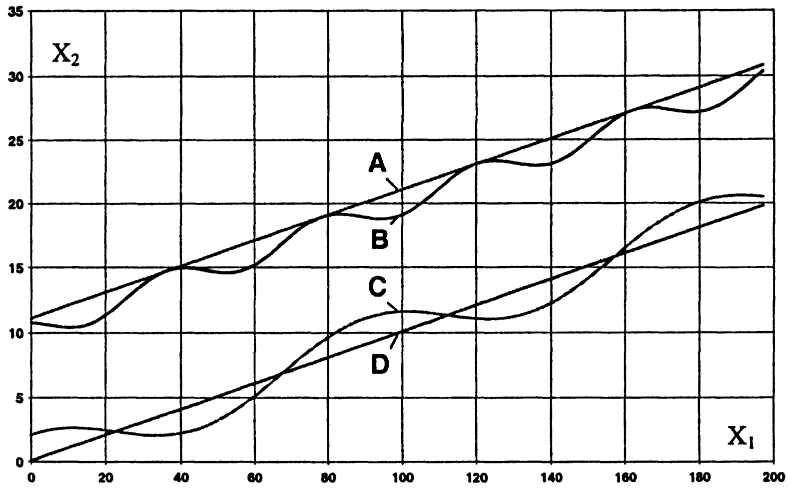


Figure 13–20. Structural and pointwise similarity.

ity, e.g. slope, curvature, position and values of extremal points or other information like smoothness or monotonicity (as a degree of membership of a trajectory to the set of monotone functions).

Here some examples of structural similarity are given for illustration:

(A) Slope and curvature of trajectories are relevant, but their position in the feature space is not relevant:

The functions y = x and y = 1.001*x + 100 are similar (both describe straight lines with approximately equal slope and a curvature of zero), whereas y = x and y = x + 0.001*sin(x) are not similar (despite the fact that they are much closer in terms of Euclidean distance).

This type of definition of similarity can be applied to classify e.g. shares as "decreasing" (A, E, H), "increasing" (B, D, G, J, K), "constant" (F, I) or 'fluctuating' (C), depending on the trajectories of their share prices (figure 13–21).

(B) Form of trajectories is relevant, but their size and position in the feature space are not relevant:

The unit circle (center at (0, 0) and radius 1) and the circle with center at (100, 0) and radius 17.4 are similar to degree 1, whereas the unit circle and the unit square are much less similar.

This type of definition may be applied to classify engines, using the airborne sound they emit during operation: amplitude and position of characteristic patterns change with speed, independent of the state of the engine. However, the characteristic pattern remains the same depending on the fact, whether an engine is intact or damaged (figure 13–22).

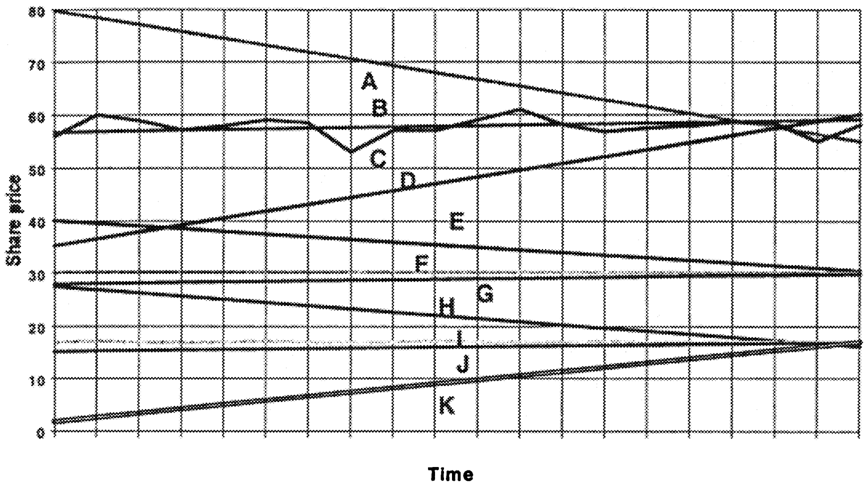


Figure 13–21. Fictitious developments of share prices.

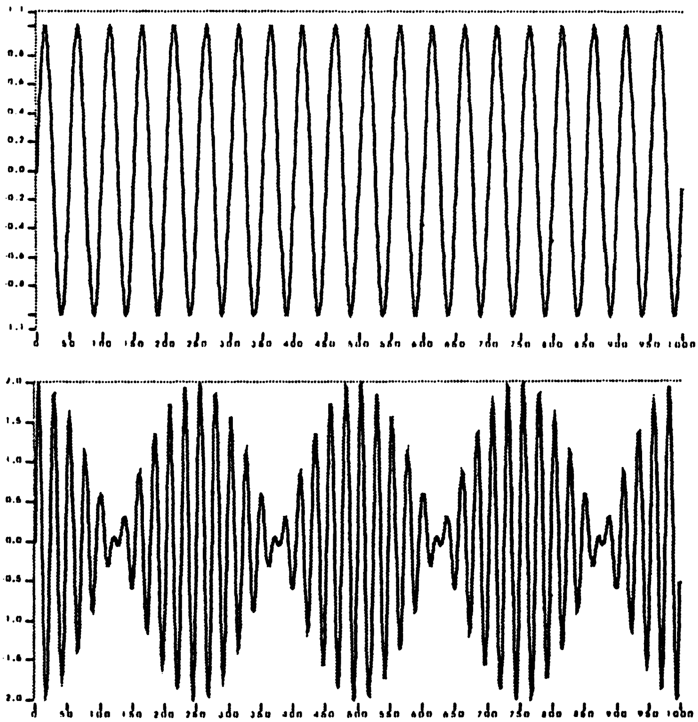


Figure 13–22. Idealized characteristic patterns of time signals for (a) an intact engine; (b) an engine with some defect.

In some cases, structural similarity can be reduced to pointwise similarity, for instance in the first example, by considering the pointwise similarity for the first and the second derivatives of the functions, respectively.

One method to define structural similarity between functions is to consider relevant characteristics of these functions (e.g. integral, extrema), which contain the information about the specific structure of functions.

The following algorithm to determine a measure for *structural similarity* between arbitrary functions f and g is proposed:

1. A set of relevant characteristics K_i, $i = 1, \ldots, m$, describing structural similarity is chosen.
2. A fuzzy set A_i labeled "admissible difference for characteristic K_i" with membership function μ_i is defined.
3. All characteristics $K_i(f)$ for the function f and $K_i(g)$ for the function g are calculated.
4. For each characteristic K_i the difference $\Delta K_i = | K_i(f) - K_i(g) |$, $i = 1, \ldots, m$, is calculated.
5. The degree of membership $s_i = \mu_i(\Delta K_i)$ of the difference ΔK_i to the fuzzy set A_i is calculated for each characteristic K_i. These membership values can be interpreted as similarities between functions f and g with respect to the chosen characteristics.
6. Finally the vector $[s_1, s_2, \ldots, s_m]$ of partial similarities is transformed using specific transformations (e.g. γ-operator, fuzzy integral, minimum, maximum) into a real number s(f, g) expressing the overall degree of similarity.

To define structural similarity between functions, the following possible characteristic values can be used:

1. Integral
2. Global minimum, maximum
3. Position of minima, maxima, zeros, inflection points
4. Number of minima, maxima, zeros, inflection points
5. Statistical characteristics
6. Parameters (if a family of parametric functions is under consideration)
7. Spline parameters (if spline approximation is used)
8. Fourier/Taylor/Wavelet coefficients
9. Range of function values
10. Median of function values
11. Center of gravity.

All these characteristic values may be calculated for the original function (trajectory) as well as for any derived function (e.g. derivatives, transformations, etc).

Furthermore, characteristics may be defined over the whole domain or just over parts of it (e.g. maximum of the first derivative in the domain $5 < x < 8$).

The definition of structural similarity as well as the choice of relevant characteristics can be simplified if the class of possible functions (trajectories) is restricted.

Pointwise Similarity between Functions. Pointwise similarity between functions is concerned with the closeness of functions in the feature space and is based on considering functional values directly (function's characteristics or derived functions are not relevant in this case). The proposed method uses similarity of the difference of two functions to the zero-function as a measure of similarity for a pair of functions. That is, similarity between functions g(x) and h(x) defined on the universe X is determined as similarity between the difference function f(x) = g(x) – h(x), $x \in X$, and the zero-function: s(g, h) = s(g–h, 0) = s(f, 0).

The following algorithm to determine a measure for *pointwise similarity* between an arbitrary function f(x) and the zero-function is proposed:

1. A fuzzy set A "approximately zero" with a membership function μ is defined (figure 13–23a). To emphasize the time focus, the variable x is taken to be time (t).
2. The degree of membership $\mu(f(x))$ of the function f(x) to the fuzzy set A is calculated for each point $x \in X$. These degrees of membership can be interpreted as (pointwise) similarities of the function f(x) to the zero-function (figure 13–23b).
3. The function $\mu(f(x))$ is eventually transformed by using specific transformations (e.g. γ-operator, fuzzy integral, minimum, maximum) into a real number s(f, 0) expressing the overall degree of being zero.

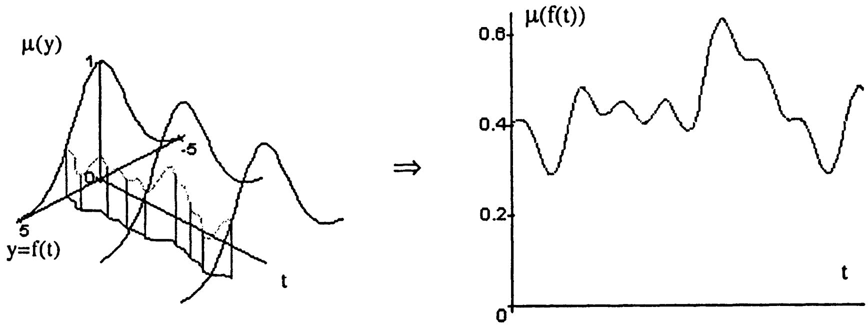


Figure 13–23. (a) The fuzzy set "approximately zero" ($\mu(y)$), the function f(t) and the resulting pointwise similarity $\mu(f(t))$; (b) projection of pointwise similarity into the plane (t, $\mu(f(t))$).

All similarity measures obtained with the help of this algorithm are invariant with respect to the addition of a function, i.e. s(g, h) = s(g + c, h + c) holds for all functions g, h and c. On the other hand, every similarity measure satisfying the above equation can be described by defining pointwise similarity between an arbitrary function and the zero-function.

The Case of Multi-dimensional Functions. The two algorithms presented above were formulated to determine structural and pointwise similarity between one-dimensional functions. The extension of these definitions for n-dimensional functions g(**x**) and h(**x**), $\mathbf{x} \in X_1 \times X_2 \times \ldots \times X_n$, is straightforward, and will be explained based on the algorithm for pointwise similarity. The modification of the algorithm can be performed in two ways:

1. Fuzzy sets A_{X_i} "approximately zero" are defined on each subuniverse X_i, i = 1, . . . , n, and the similarity measures $s_{X_i}(g,h)$, i = 1, . . . , n, are determined according to the described algorithm for projections of functions g(**x**) and h(**x**) on subuniverses. The result is the n-dimensional vector of similarities $[s_{X_1}, s_{X_2}, \ldots, s_{X_n}]$.
2. The n-dimensional fuzzy set A "approximately zero" is defined on $X_1 \times X_2 \times \ldots \times X_n$ and the similarity measure $s_{X_1 \times X_2 \ldots \times X_n}(g, h)$ is obtained for n-dimensional functions analogously to the one-dimensional case.

For some classification methods it could be desirable to transform the similarity measure into the distance measure using e.g. the relation:

$$d(g,h) = \frac{1}{s(g,h)} - 1. \tag{13.3}$$

In the first case, when n one-dimensional fuzzy sets are given, the transformation can be performed in two ways:

1. The distance measure is calculated for the components of the n-dimensional vector $[s_{X_1}, s_{X_2}, \ldots, s_{X_n}]$ resulting in the vector $[d_{X_1}, d_{X_2}, \ldots, d_{X_n}]$. The latter is then transformed into an overall distance using e.g. the Euclidean norm:

$$d(g,h) = \sqrt{\sum_{i=1,\ldots,n} d_{x_i}^2}.$$

2. The n-dimensional vector $[s_{X_1}, s_{X_2}, \ldots, s_{X_n}]$ is transformed by using some transformations (e.g. γ-operator, fuzzy integral, minimum, maximum) into an overall similarity s(g, h). Thereafter the distance measure is calculated e.g. by (13.3).

The obtained distance measure between n-dimensional functions g and h can be used as a criterion within classical methods for data analysis, allowing the classification of multi-dimensional trajectories. This topic will be discussed in the next section in more detail.

13.3.3 Approaches for Analysic Dynamic Systems

In the following, two different methods for the handling of dynamics within existing methods for data analysis are considered:

a) During preprocessing: feature vectors containing trajectories are preprocessed as to become valid inputs for classical methods such as e.g. fuzzy c-means;
b) Within the data analysis methods: classical methods are modified, so that they can process feature vectors containing trajectories directly.

Since the modifications of the classical methods do not directly affect the way clusters are built, the resulting methods are basically static. But they are suited to process dynamic objects. Each approach is handled separately in the next two sections.

Handling of Trajectories during Preprocessing. The goal of preprocessing is the preparation and representation of the measured data in order to make the classification possible and improve classification results [Famili et al. 1997]. In many data analysis tools, methods for preprocessing are integrated [MIT Data Engine 2.1 Manual 1997]. These methods include transformations of data such as calculation of the power spectrum from the time signal, computation of different characteristics or scaling / standardization of the data. Thus, usually preprocessing is performed along with feature selection.

The easiest way to integrate dynamic features into existing methods for static data analysis is to transform trajectories into real numbers (characteristic values) and to use the latter instead of the original trajectories, i.e. vector valued features are replaced by one or more real numbers. This leads to conventional feature vectors, which can be processed by classical methods. This idea is illustrated in figure 13–24, where $X_1, X_2, \ldots, X_N$ denote features represented as trajectories or vectors and $C_i(X_j)$, $i = 1, \ldots, L_j$, $j = 1, \ldots, N$, is the i-th characteristic value for feature j.

It should be noted that the number L_j, $j = 1, \ldots, N$, and type of characteristic values can vary for different features. Since this approach does not require any modifications of the classification methods used, it can very easily be used in conjunction with different methods for data analysis. The following approach requires a modification of the classification methods, but does not use any characteristic values.

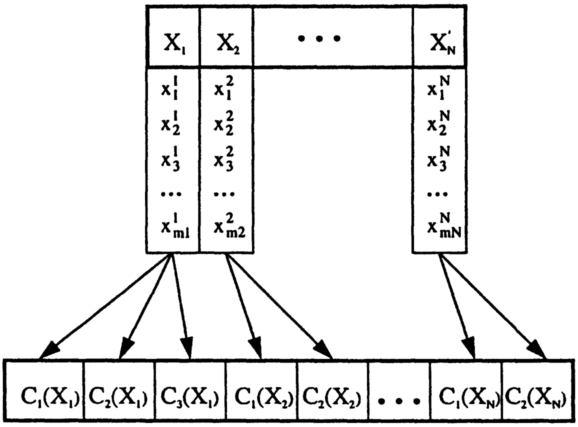


Figure 13–24. Transformation of a feature vector containing trajectories into a usual feature vector.

Handling of Trajectories within Data Analysis Methods. In the previous section, the problem of using trajectories is circumvented by reducing each trajectory to a vector of characteristic values.

In the following, another approach to handle dynamics is proposed, which is based on similarity between functions. First, some basic remarks related to the notions of distance and similarity are given.

Many data analysis methods (e.g. fuzzy c-means [Bezdek 1981], possibilistic c-means [Krishnapuran and Keller 1993], (fuzzy-) Kohonen networks [Rumelhart and McClelland 1988]) use the distance between pairs of feature vectors describing objects as a measure of similarity between these objects. Starting with a distance d(g,h) between objects, a similarity relation can be defined by $s(g, h) = 1/(1 + d(g,h))$ [Bandemer and Näther 1992]. Conversely, each strictly positive similarity relation defines a distance measure $d(g, h) = 1/s(g, h) - 1$.

All data analysis methods mentioned above use nothing else but the distance between objects and class representatives to calculate degrees of membership of objects to classes. The positions of objects in the feature space are used to determine representatives of each class. Therefore, it is sufficient to provide a distance for pairs of objects and / or class representatives to be able to calculate degrees of class membership. These considerations were used to develop a modified version of the fuzzy c-means algorithm, which is called the functional fuzzy c-means (FFCM) and is able to classify dynamic objects (i.e. objects described by trajectories). Since the features are trajectories, the class centers calculated by the

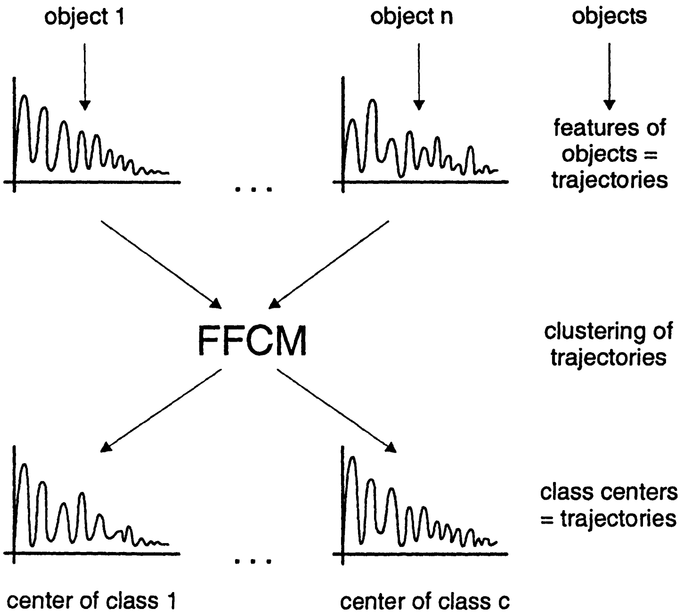


Figure 13–25. Input and output of the functional fuzzy c-means.

FFCM are not just points in the feature space, as in classical fuzzy c-means, but consist themselves of trajectories. This idea is illustrated in figure 13–25, where for the sake of simplicity objects are represented by only one feature.

The functional fuzzy c-means algorithm (FFCM) is very similar to the standard fuzzy c-means (FCM). In the following we present the FFCM and point at the differences to the FCM.

The problem of finding fuzzy clusters of trajectories in the feature space can be formulated as the minimization of an objective function $J(B, U;X)$ of the form

$$J(B, U; X) = \sum_{i=1}^{c} \sum_{j=1}^{N} (\mu_{ij})^m d^2(x_j, b_i)$$

with the following parameters

c = number of clusters
N = number of objects
m = fuzzifier (weighting exponent)
μ_{ij} = degree of membership of object j to class i
$d^2(x_j, b_i)$ = distance between object j and the class center of class i

x_j = feature vector describing object j
b_i = class center of class i

It should be noted that in the case of the FFCM the components of the feature vector of object x_j and of class center b_i are trajectories in the feature space. The distance measure is used for the calculation of $d^2(x_j, b_i)$.

The algorithm for solving the described problem consists of the following steps:

1. Initialization
   Generate values μ_{ij} for i = 1, . . . , c and j = 1, . . . , N such that

$$\sum_{i=1}^{c} \mu_{ij} = 1 \quad \forall j = 1, \ldots, N$$

2. Determination of class centers b_i

$$b_i = \frac{\sum_{j=1}^{N} (\mu_{ij})^m x_j}{\sum_{j=1}^{N} (\mu_{ij})^m}, \quad i = 1, \ldots, c,$$

Remark: The product and the sum are calculated for each component of each trajectory of the feature vectors.

3. Recalculation of membership values μ_{ij}. This is the main difference between the FFCM and the FCM. The FFCM calculates the distances d_{ij} and d_{kj} using the distance measure

$$u_{ij} = \frac{1}{\sum_{k=1}^{c} \left(\frac{d_{ij}}{d_{kj}} \right)^{\frac{2}{m-1}}}, \quad i = 1, \ldots, c, j = 1, \ldots, N$$

4. Stopping criterion: There exist many possible stopping criteria. One is to repeat steps 2 to 4 until the changes in the membership values between two iterations are smaller than a fixed threshold.

Examples of applying the FFCM to managerial and to engineering problems will be shown in chapter 15.

13.4 Tools for Fuzzy Data Analysis

13.4.1 Requirements for FDA Tools

In section 13.2, three classes of methods, primarily for classifier design and classification, were described in various degrees of detail. Each of these classes contains numerous methods, the suitability of which depends on the structure of the problem to be solved. In addition, and not described here, one needs methods for feature analysis such as fuzzy regression analysis, fuzzy discriminant analysis, etc. (for more details, see, for example, Bezdek and Pal [1992]). In other words, the tools needed for FDA are much more heterogeneous than those needed for fuzzy control as described in chapter 11.

One of the most serious problems is that very often one only knows which tool is the most suitable one after the problem has been solved. Only general guidelines are known, such as: If the shape and the number of patterns one is looking for is known, then an appropriate cluster method might best be employed. If the knowledge is available as expert knowledge but not mathematically, then a knowledge-based approach might be the best. And if this information is hidden in a large mass of available data, then an ANN might be trainable to solve the problem.

The only possible way, then, to perform FDA efficiently is to have a variety of methods readily available on a computer in order to find out by an intelligent trial-and-error method which of the methods is best suited to a specific case. This approach, however, amounts to having case tools similar to those already described for fuzzy control in chapter 11. There are only two differences: (1) Instead of only a shell for knowledge-based inference, now the methods of all three groups described in section 13.2. have to be induced, and (2) since the input data themselves are often the object of analysis and since they often are not in a suitable form to be analyzed, methods for data preprocessing also have to be included.

Data Preprocessing. If, for example, in quality control some acoustic signals have to be investigated, it becomes necessary to filter these data in order to overcome the problems of noisy input. In addition to these filter methods, some transformations of the measured data such as, for example, fast Fourier transformation (FFT) could improve the respective results. Both filter methods and FFT belong to the class of signal processing techniques. Data preprocessing includes signal processing and also conventional statistical methods.

Statistical approaches could be used to detect relationships within a data set describing a special kind of application. Here correlation analysis, regression analysis, and discrimination analysis can be applied adequately. These methods could be used, for example, to facilitate the process of feature extraction. If, say, two features from the set of available features are highly correlated, it could be sufficient for a classification to consider just one of these.

The differences between an FC tool and an FDA tool are probably responsible for the fact that hardly any FDA tools are yet available on the market. In the following section, we briefly describe the only one known so far.

13.4.2 DataEngine

DataEngine is a software tool that contains methods for data analysis described above (see figure 13–26). In particular, the combination of signal processing, statistical analysis, and intelligent systems for classifier design and classification, leads to a powerful software tool that can be used in a very broad range of applications.

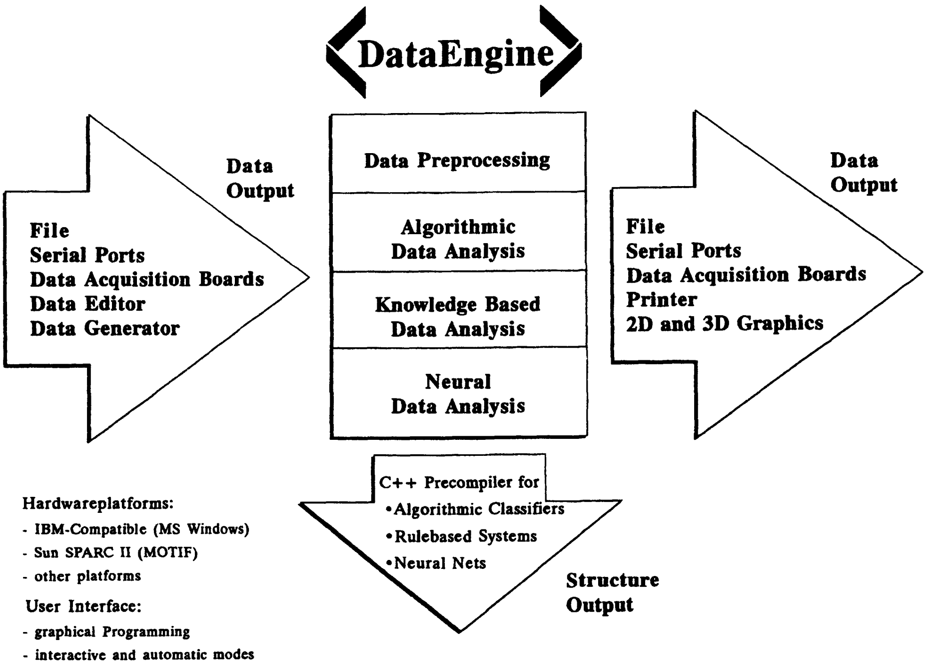


Figure 13–26. Structure of DataEngine.

DataEngine is written in an object-oriented concept in C++ and runs on all usual hardware platforms. Interactive and automatic operation supported by an efficient and comfortable graphical user interface facilitates the application of data analysis methods. In general, such applications are performed in the following three steps:

1. **Modeling a specific application with DataEngine.** Each subtask in an overall data analysis application is represented by a so-called function block in DataEngine. Such function blocks represent software modules that are specified by their input interfaces, output interfaces, and function. Examples include a certain filter method or a specific cluster algorithm. Function blocks could also be hardware modules such as neural network accelerator boards. This leads to a very high performance in time-critical applications.
2. **Classifier design (off-line data analysis).** After having modeled the application in DataEngine, off-line analysis has to be performed with given data sets to design the classifier. This task is done without process integration.
3. **Classification.** Once the classifier design is finished, the classification of new objects can be executed. Depending on specific requirements, this step can be performed in an on-line or off-line mode. If data analysis is used for decision support (e.g., in diagnosis or evaluation tasks), objects are classified off-line. Data analysis could also be applied to process monitoring and other problems where on-line classification is crucial. In such cases, direct process integration is possible by the configuration of function blocks for hardware interfaces (see figure 13–27).

DataEngine provides the following models for intelligent data analysis:

- *Fuzzy Rule Base*
  Fuzzy Rule Bases allow the representation of linguistic human knowledge in a computer. The fuzzy inference procedure is able to reproduce human decision behavior. Applications are knowledge-based diagnosis, classification tasks, control, and process modeling. Especially for data analysis tasks the DataEngine implementation offers a multistage inference procedure as well as the ability to work with symbolic variables, too.
- *Multilayer Perceptron*
  The multilayer perceptron is a supervised learning neural network. Applications are classification tasks, process modeling and control. In addition to the backpropagation learning rule with momentum and decay, DataEngine provides the quickpropagation learning rule. A configurable learning rate decay is implemented to avoid the overfitting of the neural network. The integrated pruning algorithm supports finding the optimal network architecture.

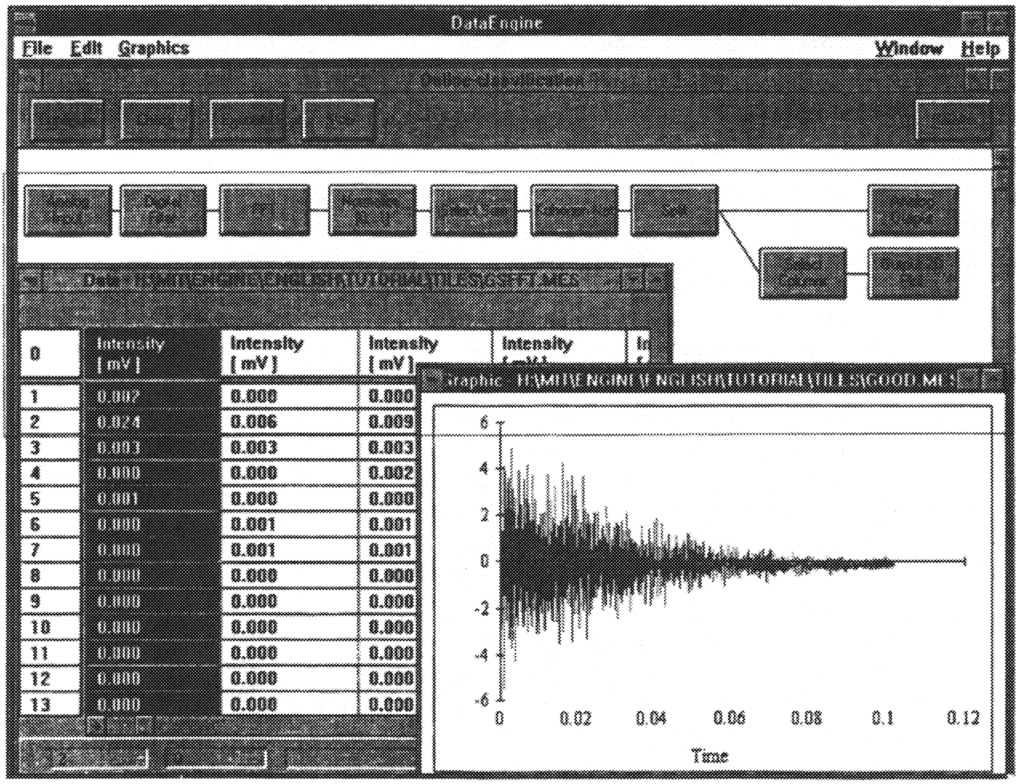


Figure 13–27. Screen shot of DataEngine.

- *Kohonen Feature Map*
  The self-organizing feature map of Kohonen is a unsupervised learning neural network, which learns the structures inside the presented data. Applications of this neural network are classification tasks and knowledge discovery. Especially for classification tasks DataEngine provides an example-based labeling algorithm, knowledge discovery is supported by the graphical visualization of the feature map.
- *Fuzzy C-Means*
  The fuzzy c-means algorithm [Bezdek 1981] is a fuzzy clustering algorithm. Its applications are clustering and classification tasks. Especially for classification tasks DataEngine provides an example-based labeling algorithm.

- *Fuzzy Kohonen Network*
  The Fuzzy Kohonen Network is a synthesis of Kohonen's feature maps and the fuzzy c-means algorithm. The use of a slightly modified fuzzy c-means inside the training algorithm of the network dramatically reduces training times. Applications of this method are clustering and classification tasks.

Each of these methods comes along with its own specialized editor. The editors offer simple and fast access to all parameters of the model and the model state can be visualized in several specialized views. All editors are structured similarly so that the training period for a new method is reasonably short.

In addition to the provided models DataEngine supplies signal processing functions such as the fast fourier transformation, smoothing and digital filtering, statistical and mathematical functions as well as a spreadsheet-based data editor support data preprocessing. The so-called cards represent a graphical macro language that can be used for the automation of tasks carried out repeatedly. DataEngine 2.1 is fully integrated into the Microsoft Windows environment and thus provides features like data exchange via the clipboard and makes full use of the Microsoft Windows printing capabilities.

The software package is extendible by so called user-defined function blocks. A user-defined function block is a special Microsoft Windows DLL (Dynamic Link Libraries) which has to conform to the DataEngine PlugIn interface.

There are three third party plug-ins available for DataEngine, which use the interface described in the previous section. Find here a short description of these products:

- *FeatureSelector PlugIn*
  The FeatureSelector PlugIn is a tool for automatic feature selection in case of classification tasks. Given a number of examples, the FeatureSelector searches for the most significant set of features which solve your classification task. For the best solutions the tool generates appropriate training data files for DataEngine.
- *Advanced Clustering Library PlugIn*
  The Advanced Clustering Library PlugIn provides nine additional clustering algorithms for DataEngine. The package contains the clustering algorithms Gustafson-Kessel, Gath-and-Geva and Fuzzy C-Means, which are implemented in several variations (probabilistic, possibilistic, parallel to axis).

13.5 Applications of FDA

Applications of data analysis are abound. Recently, fuzzy data analysis of various kinds has been applied to character recognition [Shao and Wu 1990], intelligence [Guo and Zhang 1990], market segmentation, and many other areas. Here, two applications shall be described in which the tool described above has been used.

13.5.1 Maintenance Management in Petrochemical Plants

Problem Formulation. Over 97% of the worldwide annual commercial production of ethylene is based on thermal cracking of petroleum hydrocarbons with steam. This process is commonly called pyrolysis or steam cracking. Naphtha, which is obtained by the distillation of crude oil, is the principal raw ethylene material. Boiling ranges, densities, and compositions of naphtha depend on crude oil quality.

Naphtha is heated in cracking furnaces up to 820°C–840°C, where the chemical reaction starts. The residence time of the gas stream in the furnace is determined by the severity of the cracking process. The residence time for low severity is about 1s and for high severity 0.5s. The severity of the cracking process specifies the product distribution. With high-severity cracking, the amount of ethylene in the product stream is increased and the amount of propylene is decreased significantly.

During the cracking process, acetylenic, diolefenic, and aromatic compounds are also produced, which are known to deposit coke on the inside surfaces of the furnace tubes. This coke layer inhibits heat transfer from the tube to the process gas, and therefore at some time the furnace must be shut down to remove the coke. To guarantee a continuous run of the whole plant, several furnaces are parallel integrated into the production process. The crude on-line measured process data is not suitable for determining the degree of coking. About 20 different measurements of different indicators, such as temperatures, pressures, or flows, are taken every minute. On the basis of these data only, it is not possible for the operator to decide whether the furnace is coked or not. His or her experience and the running time of the regarded furnace is the basis for this decision.

Solution by Data Analysis. Clustering methods compress the information in data sets by finding classes that can be used for classification. Similar objects are assigned to the same class. In the present case, "objects" are different states of a cracking furnace during a production period. Objects are described by different features. Features are the on-line measured quantities, such as temperatures, etc.

Figure 13–28 shows the structure of the cracking furnace under consideration. Features describing the process are primarily temperatures and flows. The classes are “coked state” and “decoked state.” Fuzzy cluster methods were used to determine the coking of 10 cracking furnaces of a thermal cracker. The data of one year have been analyzed. The process of coking lasts about 60 days. Therefore only mean values of a day of the measured quantities were considered. For different furnaces, the centers of coked and decoked classes were found by searching for coked and decoked states in the data set. Figure 13–29 shows the temperature profile of a furnace during the whole year. Characteristic peaks, where temperature decreases significantly, result from decoking processes. K1 and K2 describe decoked and coked states of the furnace.

The temperature profile shows no characteristic shape that results from coking. Furnace temperature is only one of the features shown in figure 13–29. There are dependencies between features, so a determination of coking through consideration of only the feature “temperature” is not possible.

Figure 13–30 shows the membership values of a furnace state during a production period using the classifier. The values describe the membership of the current furnace state in the coked class. The membership values increase continuously and reach nearly 1 at the end of the production period.

The classifier works on-line and classifies the current furnace state with reference to the coking problem. The operator can use this information to check how long the furnace under consideration will be able to run until it has to be decoked. As a result, it becomes easier to make arrangements concerning logistical questions, e.g., ordering the correct amounts of raw material or not being understaffed at certain times.

13.5.2 Acoustic Quality Control

In acoustic quality control, many efforts have been undertaken to automate the respective control tasks that are usually performed by humans.

Even if there are many computerized systems for automatic quality control via analysis of acoustic signals, some of the problems cannot be solved adequately yet. Below, an example of acoustic control of ceramic goods is presented to show the potentials of fuzzy data analysis in this respect.

Problem Formulation. In cooperation with a producer of tiles, a prototype has been built that shows the potentials of automatic quality control. At this point, an employee of this company has to check the quality of the final product by hitting it with a hammer and deciding about the quality of the tile based on the

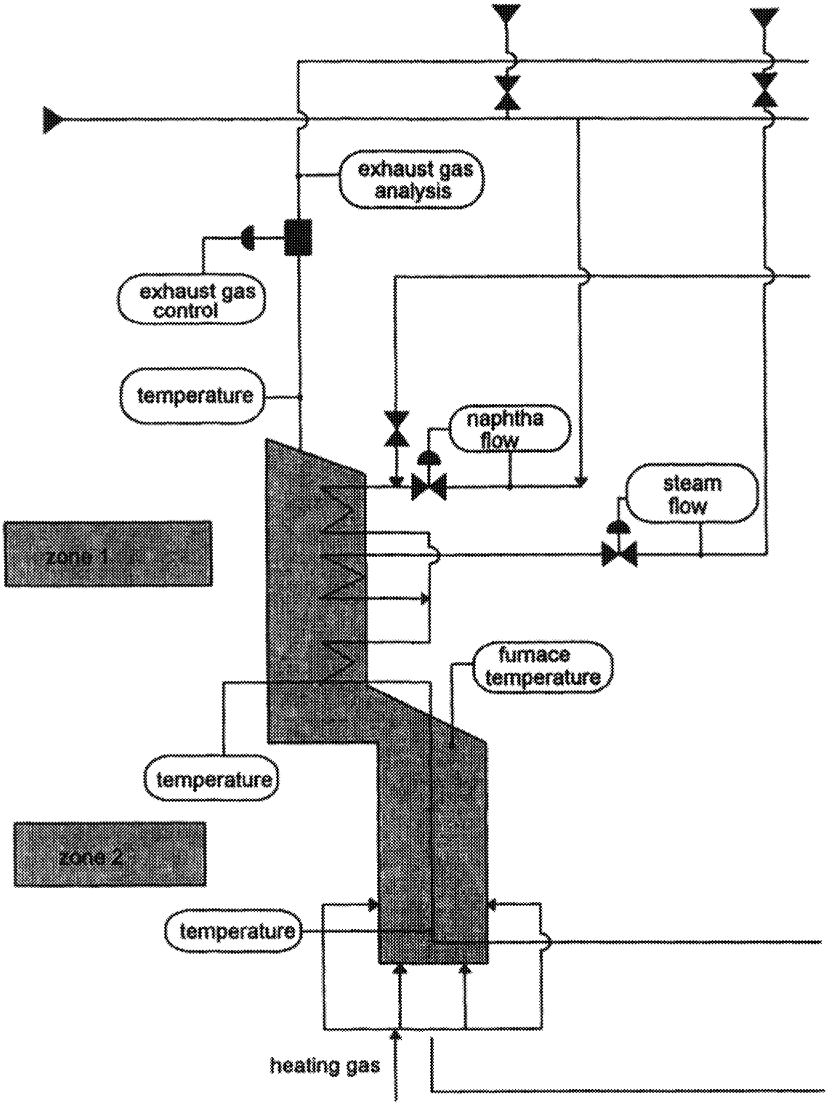


Figure 13–28. Cracking furnace.

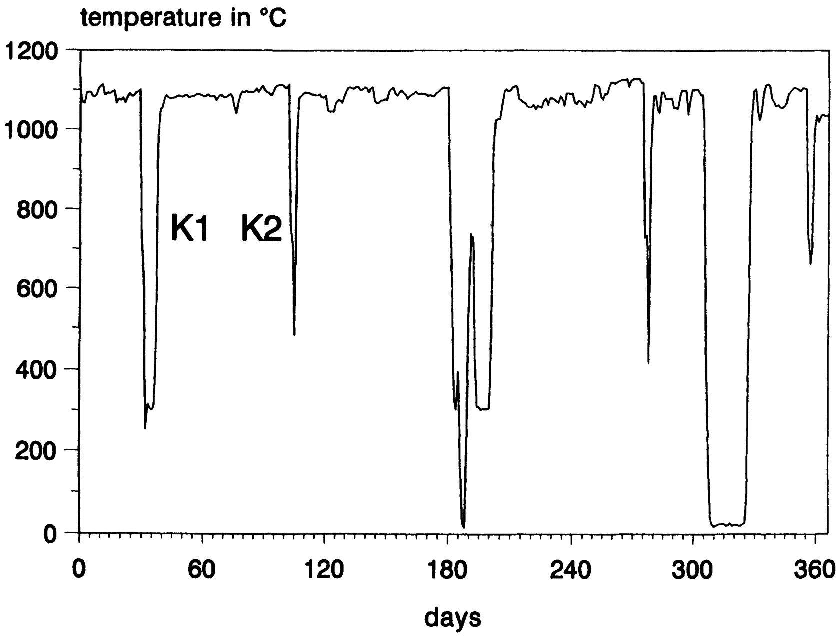


Figure 13–29. Furnace temperature.

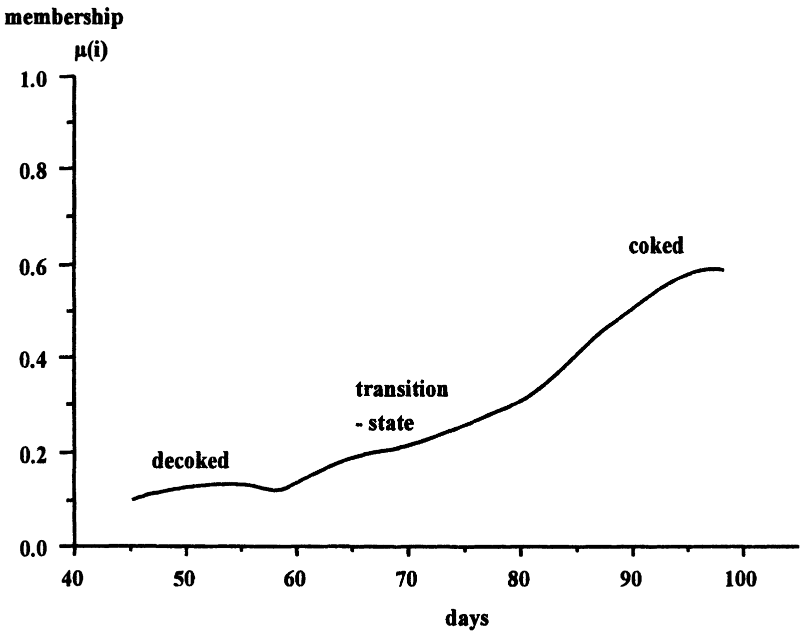
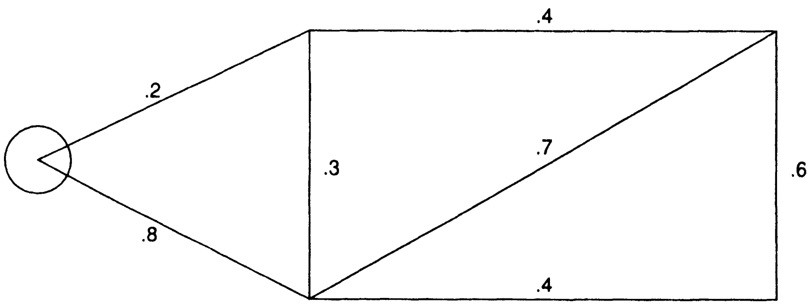


Figure 13–30. Fuzzy classification of a continuous process.

resulting sound. Since cracks in the tile cause an unusual sound, an experienced worker can distinguish between good and bad tiles.

Solution Process. In this application, algorithmic methods for classifier design and classification were used to detect cracks in tiles. In the experiments, the tiles are hit automatically, and the resulting sound is recorded via a microphone and an A/D-converter.

Then signal processing methods like filtering and fast Fourier transformations (FFI) transform these sound data into a spectrum that can be analyzed. For example, the time signal is transformed by an FFT into the frequency spectrum. From this frequency spectrum, several characteristic features are extracted that could be used to distinguish between good and bad tiles. The feature values are the sum of amplitude values in some specified frequency intervals. In the experiments, a six-dimensional feature vector showed best results. After this feature extraction, the fuzzy c-means algorithm found fuzzy classes that could be interpreted as good and bad tiles. Since a crisp distinction between these two classes is not always possible, fuzzy cluster techniques have an advantage: not only do they distinguish bad from good tiles but also the intermediate qualities can be defined (see figure 13–31).



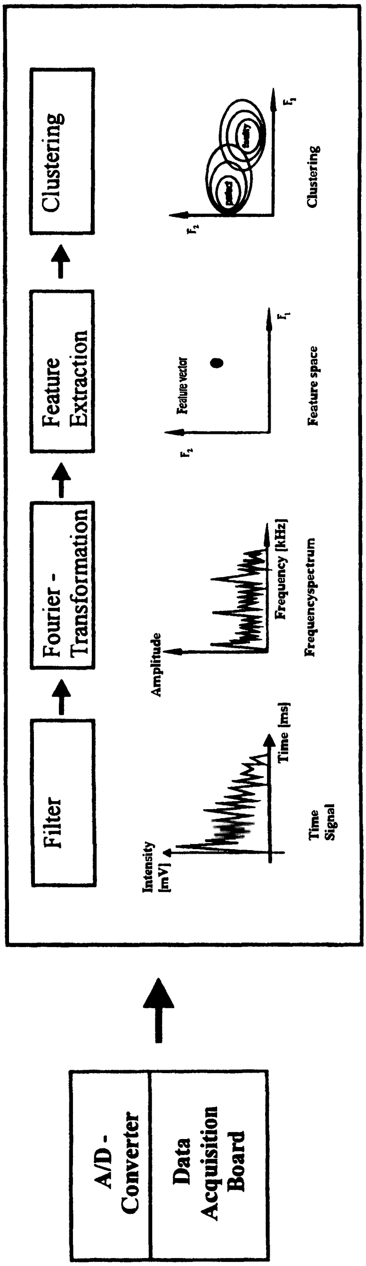


Figure 13–31. Application of DataEngine for acoustic quality control.

Exercises

1. Describe three example problems from the areas of engineering and management, each of which can be considered as a problem of pattern recognition.
2. How is the dimensionality of the data space reduced in pattern recognition?
3. What is the center of a cluster and how can it be defined?
4. Which basic types of objective-function algorithms exist in cluster analysis?
5. Consider the following fuzzy graph:
   Determine the clusters of the graph in dependence of the τ-degree (cf. figure 13–6).
6. Let $X = \{x_1, x_2, x_3, x_4\}$ and let each x_i be a point in three-dimensional space. Determine all 3-partitions that are possible and display them as shown in example 13-1.
7. Give three possible fuzzy three-partitions for the problem given in exercise 6.
8. Let $X = \{(1, 1), (1, 3), (10, 1), (10, 3), (5, 2)\}$ be a set of points in the plane. Determine a crisp 3-partition that groups together $(1, 3)$ and $(10, 3)$ and that minimizes the Euclidean norm metric. Do the same for the variance criterion.
9. Determine the cluster validity of the clusters shown in figures 13–11 and 13–12 by computing the partition coefficient and the partition entropy.

14 DECISION MAKING IN FUZZY ENVIRONMENTS

14.1 Fuzzy Decisions

The term *decision* can have very many different meanings, depending on whether it is used by a lawyer, a businessman, a general, a psychologist, or a statistician. In one case it might be a legal construct, and in another a mathematical model; it might also be a behavioral action or a specific kind of information processing. While some notions of a "decision" have a formal character, others try to describe decision making in reality.

In classical (normative, statistical) decision theory, a decision can be characterized by a set of decision alternatives (the decision space); a set of states of nature (the state space); a relation assigning to each pair of a decision and state a result; and finally, the utility function that orders the results according to their desirability. When deciding under certainty, the decision maker knows which state to expect and chooses the decision alternative with the highest utility, given the prevailing state of nature. When deciding under risk, he does not know exactly which state will occur; he only knows a probability function of the states. Then decision making becomes more difficult. We shall restrict our attention to decision making under certainty. In this instance, the model of decision making is nonsymmetric in the following sense: The decision space can be described

H.-J. Zimmermann, *Fuzzy Set Theory - and Its Applications*

either by enumeration or by a number of constraints. The utility function orders the decision space via the one-to-one relationship of results to decision alternatives. Hence we can only have *one* utility function, supplying the order, but we may have several constraints defining the decision space.

Example 14–1

Let us assume that the board of directors wants to determine the optimal dividend. Their *objective function* (utility function) is to maximize the dividend. The *constraint* defining the decision space is that the dividend be between zero and 6%. Hence the optimal dividend is "Between 0 and 6%" *and* "maximal." (The constraint does *not* impose an order on the decision space!) The optimal dividend will obviously be 6%. Assigning a linear utility function, figure 14–1 illustrates these relationships.

In 1970 Bellman and Zadeh considered this classical model of a decision and suggested a model for decision making in a fuzzy environment that has served as a point of departure for most of the authors in "fuzzy" decision theory. They consider a situation of decision making under certainty, in which the objective function as well as the constraint(s) are fuzzy, and argue as follows: The fuzzy objective function is characterized by its membership function, and so are the constraints. Since we want to satisfy (optimize) the objective function as well as the constraints, a decision in a fuzzy environment is defined by analogy to nonfuzzy environments as the selection of activities that simultaneously satisfy

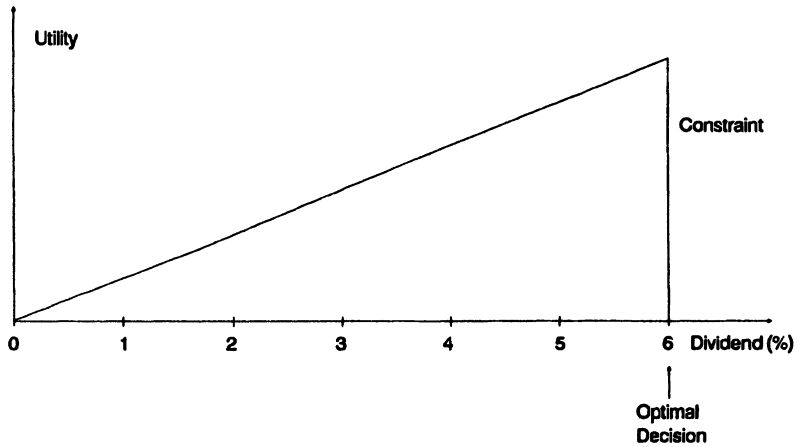


Figure 14–1. A classical decision under certainty.

objective function(s) *and* constraints. According to the above definition and assuming that the constraints are "noninteractive," the logical "and" corresponds to the intersection. The "decision" in a fuzzy environment can therefore be viewed as the intersection of fuzzy constraints and fuzzy objective function(s). The relationship between constraints and objective functions in a fuzzy environment is therefore fully symmetric, that is, there is no longer a difference between the former and the latter.

This concept is illustrated by the following example [Bellman and Zadeh 1970, B-148]:

Example 14–2

Objective function "x should be substantially larger than 10," characterized by the membership function

$$\mu_{\tilde{O}}(x) = \begin{cases} 0 & x \leq 10 \\ \left(1 + (x-10)^{-2}\right)^{-1} & x > 10 \end{cases}$$

Constraint "x should be in the vicinity of 11," characterized by the membership function

$$\mu_{\tilde{C}}(x) = \left(1 + (x-11)^4\right)^{-1}$$

The membership function $\mu_{\tilde{D}}(x)$ of the decision is then

$$\mu_{\tilde{D}}(x) = \mu_{\tilde{O}}(x) \wedge \mu_{\tilde{C}}(x)$$

$$\mu_{\tilde{D}}(x) = \begin{cases} \min\left\{\left(1 + (x-10)^{-2}\right)^{-1}, \left(1 + (x-11)^4\right)^{-1}\right\} & \text{for } x > 10 \\ 0 & \text{for } x \leq 10 \end{cases}$$

$$= \begin{cases} \left(1 + (x-11)^4\right)^{-1} & \text{for } x > 11.75 \\ 0 & \text{for } 10 < x \leq 11.75 \\ & \text{for } x \leq 10 \end{cases}$$

This relation is depicted in figure 14–2. Let us now modify example 14–1 accordingly.

Example 14–3

The board of directors is trying to find the "optimal" dividend to be paid to the shareholders. For financial reasons this dividend ought to be attractive, and for reasons of wage negotiations it should be modest.

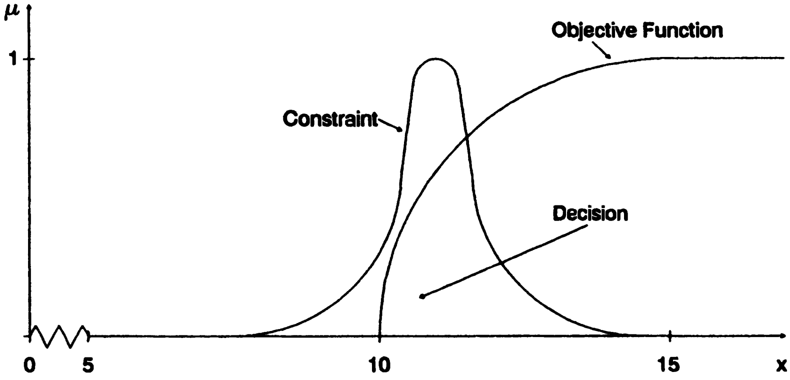


Figure 14–2. A fuzzy decision.

The fuzzy set of the objective function "attractive dividend" could, for instance, be defined by:

$$\mu_{\tilde{O}}(x) = \begin{cases} 1 & x \geq 5.8 \\ \dfrac{1}{2{,}100}[-29x^3 - 366x^2 - 877x + 540] & 1 < x < 5.8 \\ 0 & x \leq 1 \end{cases}$$

The fuzzy set (constraint) "modest dividend" could be represented by

$$\mu_{\tilde{C}}(x) = \begin{cases} 1 & x \leq 1.2 \\ \dfrac{1}{2{,}100}[-29x^3 - 243x^2 + 16x + 2{,}388] & 1.2 < x < 6 \\ 0 & x \geq 6 \end{cases}$$

The fuzzy set "decision" is then characterized by its membership function

$$\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{O}}(x), \mu_{\tilde{C}}(x)\}$$

If the decision maker wants to have a "crisp" decision proposal, it seems appropriate to suggest the dividend with the highest degree of membership in the fuzzy set "decision." Let us call this "maximizing decision," defined by

$$x_{\max} = \arg\left(\max_x \min\ \{\mu_{\tilde{O}}(x), \mu_{\tilde{C}}(x)\}\right)$$

Figure 14–3 sketches this situation.

After these introductory remarks and examples, we shall formally define a decision in a fuzzy environment in the sense of Bellman and Zadeh.

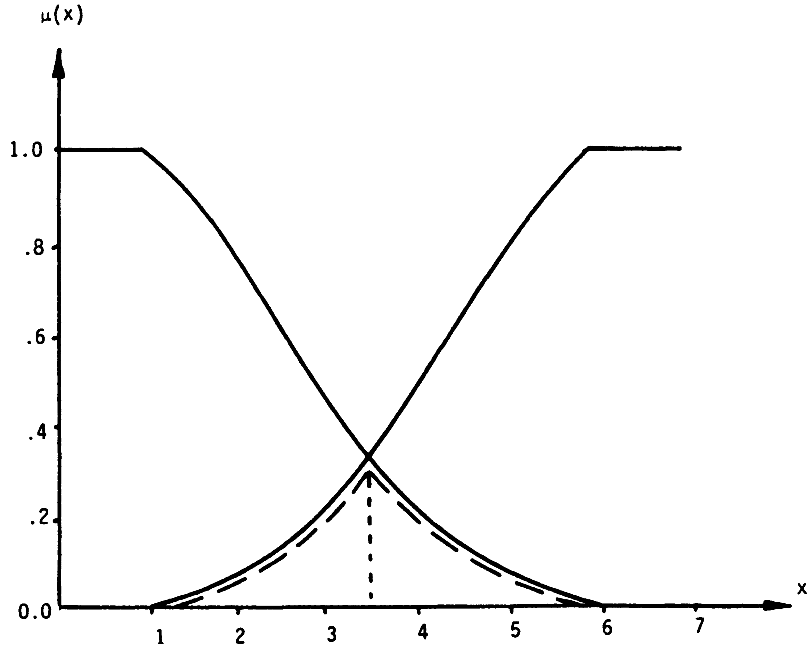


Figure 14–3. Optimal dividend as maximizing decision.

Definition 14–1 [Bellman and Zadeh 1970, B-148]

Assume that we are given a fuzzy goal $\tilde{G}$ and a fuzzy constraint $\tilde{C}$ in a space of alternatives X. Then $\tilde{G}$ and $\tilde{C}$ combine to form a *decision*, $\tilde{D}$, which is a fuzzy set resulting from intersection of $\tilde{G}$ and $\tilde{C}$. In symbols, $\tilde{D} = \tilde{G} \cap \tilde{C}$, and correspondingly,

$$\mu_{\tilde{D}} = \min \{\mu_{\tilde{G}}, \mu_{\tilde{C}}\}$$

More generally, suppose that we have n goals $\tilde{G}_1, \ldots, \tilde{G}_n$ and m constraints $\tilde{C}_1, \ldots, \tilde{C}_m$. Then the resultant decision is the intersection of the given goals $\tilde{G}_1, \ldots, \tilde{G}_n$ and the given constraints $\tilde{C}_1, \ldots, \tilde{C}_m$. That is,

$$\tilde{D} = \tilde{G}_1 \cap \tilde{G}_2 \cap \ldots \cap \tilde{G}_n \cap \tilde{C}_1 \cap \tilde{C}_2 \cap \ldots \cap \tilde{C}_m$$

and correspondingly

$$\begin{aligned} \mu_{\tilde{D}} &= \min \{\mu_{\tilde{G}_1}, \mu_{\tilde{G}_2}, \ldots, \mu_{\tilde{G}_n}, \mu_{\tilde{C}_1}, \mu_{\tilde{C}_2}, \ldots, \mu_{\tilde{C}_m}\} \\ &= \min \{\mu_{\tilde{G}_i}, \mu_{\tilde{C}_j}\} = \min \{\mu_i\} \end{aligned}$$

Definition 14–1 implies essentially three assumptions:

1. The "and" connecting goals and constraints in the model corresponds to the "logical and."
2. The logical "and" corresponds to the set-theoretic intersection.
3. The intersection of fuzzy sets is defined in the possibilistic sense by the min-operator.

Bellman and Zadeh indicated in their 1970 paper that the min-interpretation of the intersection might have to be modified depending on the context. "In short, a broad definition of the concept of decision may be stated as: Decision = Confluence of Goals in Constraints" [Bellman and Zadeh 1970, B-149].

The question arises whether even the intersection interpretation is a generally acceptable assumption or whether "confluence" has to be interpreted in an even more general way. Let us consider the following example.

Example 14–4

An instructor at a university must decide how to grade written test papers. Let us assume that the problem to be solved in the test was a linear programming problem and that the student was free to solve it either graphically or using the simplex method. The student has done both. The student's performance is expressed—for graphical solution as well as for the algebraic solution—as the achieved degree of membership in the fuzzy sets "good graphical solution" ($\tilde{G}$) and "good simplex solution" ($\tilde{S}$), respectively. Let us assume that he reaches

$$\mu_{\tilde{G}} = 0.9 \quad \text{and} \quad \mu_{\tilde{S}} = 0.7$$

If the grade to be awarded by the instructor corresponds to the degree of membership of the fuzzy set "good solutions of linear programming problems" it would be quite conceivable that his grade $\mu_{\widetilde{LP}}$ could be determined by

$$\mu_{\widetilde{LP}} = \max\{\mu_{\tilde{G}}, \mu_{\tilde{S}}\} = \max\{0.9, 0.7\} = 0.9$$

The two definitions of decisions—as the intersection or the union of fuzzy sets—imply essentially the following: The interpretation of a decision as the intersection of fuzzy sets implies no positive compensation (trade-off) between the degrees of membership of the fuzzy sets in question, if either the minimum or the product is used as an operator. Each of them yields a degree of membership of the resulting fuzzy set (decision), which is on or below the lowest degree of membership of all intersecting fuzzy sets (see example 14–3).

The interpretation of a decision as the union of fuzzy sets, using the max-operator, leads to the maximum degree of membership achieved by any of the fuzzy sets representing objectives or constraints. This amounts to a full compensation of lower degrees of membership by the maximum degree of membership (see example 14–4).

Observing managerial decisions, one finds that there are hardly any decisions with no compensation between either different degrees of goal achievement or the degrees to which restrictions are limiting the scope of decisions. The compensation, however, rarely ever seems to be "complete," as would be assumed using the max-operator. It may be argued that compensatory tendencies in human aggregation are responsible for the failure of some classical operators (min, product, max) in empirical investigations.

Two conclusions can probably be drawn: Neither the noncompensatory "and" represented by operators that map between zero and the minimum degree of membership (min-operator, product-operator, Hamacher's conjunction operator [definition 3–15], Yager's conjunction operator [definition 3–16]) nor the fully compensatory "or" represented by the operators that map between the maximum degree of membership and 1 (maximum, algebraic sum, Hamacher's disjunction operator, Yager's disjunction operator) are appropriate to model the aggregation of fuzzy sets representing managerial decisions.

"Confluence of Goals and Constraints" should therefore be interpreted as in definition 14–2.

Definition 14–2

Let $\mu_{\tilde{G}_i}(x)$, $i = 1, \ldots, m$, $x \in X$, be membership functions of constraints, defining the decision space, and let $\mu_{\tilde{G}_j}(x)$, $j = 1, \ldots, n$, $x \in X$ be the membership functions of objective (utility) functions or goals.

A *decision* is then defined by its membership function

$$\mu_{\tilde{D}}(x) = \circledast_i \mu_{\tilde{C}_i}(x) * \circledast_j \mu_{\tilde{G}_j}(x), i = 1, \ldots, m, j = 1, \ldots, n$$

where $*$, $\circledast_i$, $\circledast_j$ denote appropriate, possibly context-dependent "aggregators" (connectives).

We shall discuss the question of appropriate connectives in more detail in chapter fifteen. Before we turn to fuzzy mathematical programming, it should be mentioned that the symmetry that is a property of all definitions based on Bellman-Zadeh's concept (irrespective of the operators used) is not considered adequate by all authors (for example, see Asai et al. [1975]).

14.2 Fuzzy Linear Programming

Linear programming models shall be considered as a special kind of decision model: The decision space is defined by the constraints; the "goal" (utility function) is defined by the objective function; and the type of decision is decision making under certainty. The classical model of linear programming can be stated as

$$\begin{aligned} \text{maximize} \quad & f(x) = c^T x \\ \text{such that} \quad & Ax \leq b \\ & x \geq 0 \\ \text{with } c, x \in \mathbb{R}^n, & \ b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n} \end{aligned} \tag{14.1}$$

Let us now depart from the classical assumptions that all coefficients of A, b, and c are crisp numbers, that $\leq$ is meant in a crisp sense, and that "maximize" is a strict imperative!

If we assume that the LP-decision has to be made in fuzzy environments, quite a number of possible modifications of model (14.1) exist. First of all, the decision maker might not really want to actually maximize or minimize the objective function. Rather, he or she might want to reach some aspiration levels that might not even be definable crisply. Thus he or she might want to "improve the present cost situation considerably," and so on.

Secondly, the constraints might be vague in one of the following ways: The $\leq$ sign might not be meant in the strictly mathematical sense, but smaller violations might well be acceptable. This can happen if the constraints represent aspiration levels as mentioned above or if, for instance, the constraints represent sensory requirements (taste, color, smell, etc.) that cannot adequately be approximated by a crisp constraint. Of course, the coefficients of the vectors b or c or of the matrix A itself can have a fuzzy character either because they are fuzzy in nature or because perception of them is fuzzy.

Finally, the role of the constraints can be different from that in classical linear programming, where the violation of any single constraint by any amount renders the solution infeasable. The decision maker might accept small violations of constraints but might also attach different (crisp or fuzzy) degrees of importance to violations of different constraints. Fuzzy linear programming offers a number of ways to allow for all these types of vagueness, and we shall discuss some of them below.

First of all, one can either accept Bellman–Zadeh's concept of a symmetrical decision model (see definition 14–1) or develop specific models on the basis of a nonsymmetrical basic model of a "fuzzy" decision [Orlovsky 1980; Asai et al.

1975]. Here we shall adopt the former, more common, approach. Secondly, one has to decide how a fuzzy "maximize" is to be interpreted, or whether to stick to a crisp "maximize." In the latter case, complications arise on how to connect a crisp objective function with a fuzzy solution space. We will discuss one approach for a fuzzy goal and one approach for a crisp objective function.

Finally, one has to decide where and how fuzziness enters the constraints. Some authors [Tanaka and Asai 1984] consider the coefficients of A, b, c as fuzzy numbers and the constraints as fuzzy functions. We shall here adapt another approach that seems to be more efficient computationally and more closely resembles Bellman–Zadeh's model in definition 14–1: We shall represent the goal and the constraints by fuzzy sets and then aggregate them in order to derive a maximizing decision.

In both approaches, one also has to decide on the type of membership function characterizing either the fuzzy numbers or the fuzzy sets representing goal and constraints.

In classical LP, the "violation" of any constraint in model (14.1) renders the solution infeasible. Hence all constraints are considered to be of equal weight or importance. When departing from classical LP, this conclusion is no longer true, and one also has to worry about the relative weights attached to the constraints.

Before we develop a specific model of linear programming in a fuzzy environment, it should have become clear that in contrast to classical linear programming, "fuzzy linear programming" is *not* a uniquely defined type of model; many variations are possible, depending on the assumptions or features of the real situation to be modeled.

14.2.1 Symmetric Fuzzy LP

Let us now turn to a first basic model for "fuzzy linear programming." In model (14.1), we shall assume that the decision maker can establish an aspiration level, z, for the value of the objective function he or she wants to achieve and that each of the constraints is modeled as a fuzzy set. Our fuzzy LP then becomes:
Find x such that

$$\begin{aligned} c^T x &\gtrapprox z \\ Ax &\lessapprox b \\ x &\geq 0 \end{aligned} \tag{14.2}$$

Here $\lessapprox$ denotes the fuzzified version of $\leq$ and has the linguistic interpretation "essentially smaller than or equal to." $\gtrapprox$ denotes the fuzzified version of $\geq$ and has the linguistic interpretation "essentially greater than or equal to." The objective function in model (14.1) might have to be written as a minimizing goal in order to consider z as an upper bound.

We see that model (14.2) is fully symmetric with respect to objective function and constraints, and we want to make that even more obvious by substituting $\binom{-c}{A} = B$ and $\binom{-z}{b} = d$. Then model (14.2) becomes:
Find x such that

$$Bx \lesseqqgtr d$$
$$x \geq 0 \tag{14.3}$$

Each of the $(m + 1)$ rows of model (14.3) shall now be represented by a fuzzy set, the membership functions of which are $\mu_i(x)$. Following definition 14–1, the membership function of the fuzzy set "decision" of model (14.3) is

$$\mu_{\tilde{D}}(x) = \min_i \{\mu_i(x)\} \tag{14.4}$$

$\mu_i(x)$ can be interpreted as the degree to which x fulfills (satisfies) the fuzzy unequality $B_ix \leq d_i$ (where B_i denotes the ith row of B).

Assuming that the decision maker is interested not in a fuzzy set but in a crisp "optimal" solution, we could suggest the "maximizing solution" to equation (13.4), which is the solution to the possibly nonlinear programming problem

$$\max_{x \geq 0} \min \{\mu_i(x)\} = \max_{x \geq 0} \mu_{\tilde{D}}(x) \tag{14.5}$$

Now we have to specify the membership functions $\mu_i(x)$. $\mu_i(x)$ should be 0 if the constraints (including the objective function) are strongly violated, and 1 if they are very well satisfied (i.e., satisfied in the crisp sense); and $\mu_i(x)$ should increase monotonously from 0 to 1, that is,

$$\mu_i(x) = \begin{cases} 1 & \text{if} \quad B_ix \leq d_i \\ \in [0,1] & \text{if} \quad d_i < B_ix \leq d_i + p_i \\ 0 & \text{if} \quad B_ix > d_i + p_i \end{cases} \quad i = 1, \ldots, m+1 \tag{14.6}$$

Using the simplest type of membership function, we assume them to be linearly increasing over the "tolerance interval" p_i:

$$\mu_i(x) = \begin{cases} 1 & \text{if} \quad B_ix \leq d_i \\ 1 - \dfrac{B_ix - d_i}{p_i} & \text{if} \quad d_i < B_ix \leq d_i + p_i \\ 0 & \text{if} \quad B_ix > d_i + p_i \end{cases} \quad i = 1, \ldots, m+1 \tag{14.7}$$

The p_i are subjectively chosen constants of admissible violations of the constraints and the objective function. Substituting equation (14.7) into problem (14.5) yields, after some rearrangements [Zimmermann 1976] and with some additional assumptions,

$$\max_{x \geq 0} \min_i \left(1 - \frac{B_i x - d_i}{\mu_i}\right) \tag{14.8}$$

Introducing one new variable, λ, which corresponds essentially to equation (14.4), we arrive at

$$\begin{aligned} &\text{maximize} && \lambda \\ &\text{such that} && \lambda p_i + B_i x \leq d_i + p_i \quad i = 1, \ldots, m+1 \\ &&& x \geq 0 \end{aligned} \tag{14.9}$$

If the optimal solution to problem (14.9) is the vector (λ, x_0), then x_0 is the maximizing solution (14.5) of model (14.2), assuming membership functions as specified in (14.7).

The reader should realize that this maximizing solution can be found by solving one standard (crisp) LP with only one more variable and one more constraint than in model (14.3). Consequently, this approach is computationally very efficient.

A slightly modified version of models (14.8) and (14.9), respectively, results if the membership functions are defined as follows: A variable t_i, $i = 1, \ldots, m + 1$, $0 \leq t_i \leq p_i$, is defined that measures the degree of violation of the ith constraint: The membership function of the ith row is then

$$\mu_i(x) = 1 - \frac{t_i}{p_i} \tag{14.10}$$

The crisp equivalent model is then

$$\begin{aligned} &\text{maximize} && \lambda \\ &\text{such that} && \lambda p_i + t_i \leq p_i \quad i = 1, \ldots, m+1 \\ &&& B_i x - t_i \leq d_i \\ &&& t_i \leq p_i \\ &&& x, t \geq 0 \end{aligned} \tag{14.11}$$

This model is larger than model (14.9), even though the set of constraints $t_i \leq p_i$ is actually redundant. Model (14.11) has some advantages, however, in particular when performing sensitivity analysis, which will be discussed in the second volume on decisions in fuzzy environments.

Example 14–5

A company wanted to decide on the size and structure of its truck fleet. Four differently sized trucks (x_1 through x_4) were considered. The objective was to minimize cost, and the constraints were to supply all customers (who have a strong seasonally fluctuating demand). This meant certain quantities had to be moved (quantity constraint) and a minimum number of customers per day had to be contacted (routing constraint). For other reasons, it was required that at least six of the smallest trucks be included in the fleet. The management wanted to use quantitative analysis and agreed to the following suggested linear programming approach:

minimize

$$41{,}400x_1 + 44{,}300x_2 + 48{,}100x_3 + 49{,}100x_4$$

subject to constraints

$$\begin{aligned} 0.84x_1 + 1.44x_2 + 2.16x_3 + 2.4x_4 &\geq 170 \\ 16x_1 + 16x_2 + 16x_3 + 16x_4 &\geq 1{,}300 \\ x_1 &\geq 6 \\ x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The solution was $x_1 = 6$, $x_2 = 16.29$, $x_3 = 0$, $x_4 = 58.96$, and Min Cost = 3,864,975. When the results were presented to management, it turned out that the findings were considered acceptable but that the management would rather have some "leeway" in the constraints. Management felt that because demand forecasts had been used to formulate the constraints (and because forecasts never turn out to be correct!), there was a danger of not being able to meet higher demands by their customers.

When they were asked whether or not they really wanted to "minimize transportation cost," they answered: Now you are joking. A few months ago you told us that we have to minimize cost; otherwise, you could not model our problem. Nobody knows minimum cost anyway. The budget shows a cost figure of $4.2 million, a figure that must not be exceeded. If you want to keep your contract, you better stay considerably below this figure.

Since management felt forced into giving precise constraints (because of the model) in spite of the fact that it would rather have given some intervals, model (14.3) was selected to model the management's perceptions of the problem satisfactorily. The following parameters were estimated:

Lower bounds of the tolerance interval:

$$d_1 = 3{,}700{,}000 \quad d_2 = 170 \quad d_3 = 1{,}300 \quad d_4 = 6$$

Spreads of tolerance intervals:

$$p_1 = 500{,}000 \quad p_2 = 10 \quad p_3 = 100 \quad p_4 = 6$$

After dividing all rows by their respective p_i's and rearranging in such a way that only λ remains on the left-hand side, our problem in the form of (14.9) became:

Maximize λ subject to constraints

$$\begin{aligned} 0.083x_1 + 0.089x_2 + 0.096x_3 + 0.098x_4 + \lambda &\le 8.4 \\ 0.084x_1 + 0.144x_2 + 0.216x_3 + 0.24x_4 - \lambda &\ge 17 \\ 0.16x_1 + 0.16x_2 + 0.16x_3 + 0.16x_4 - \lambda &\ge 13 \\ 0.167x_1 - \lambda &\ge 1 \\ \lambda, x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

The solution is as Follows:

Unfuzzy	*Fuzzy*
$x_1 = 6$	$x_1 = 17.414$
$x_2 = 16.29$	$x_2 = 0$
$x_4 = 58.96$	$x_4 = 66.54$
$Z = 3{,}864{,}975$	$Z = 3{,}988{,}250$
Constraints:	
1. 170	174.33
2. 1,300	1,343.328
3. 6	17.414

As can be seen from the solution, "leeway" has been provided with respect to all constraints and at additional cost of 3.2%.

The main advantage, compared to the unfuzzy problem formulation, is the fact that the decision maker is not forced into a precise formulation because of mathematical reasons even though he or she might only be able or willing to describe the problem in fuzzy terms. Linear membership functions are obviously only a very rough approximation. Membership functions that monotonically increase or decrease, respectively, in the interval of $[d_i, d_i + p_i]$ can also be handled quite easily, as well be shown later.

So far, the objective function and all constraints were considered fuzzy. If some of the constraints are crisp, $Dx \le b$, then these constraints can easily be added to formulations (14.9) or (14.11), respectively. Thus problem (14.9) would, for instance, become:

$$\begin{aligned} &\text{maximize} && \lambda \\ &\text{such that} && \lambda p_i + B_i x \leq d_i + p_i \quad i = 1, \ldots, m+1 \\ &&& Dx \leq b \\ &&& x, \lambda \geq 0 \end{aligned} \tag{14.12}$$

Let us now turn to the case in which the objective function is crisp and the solution space is fuzzy.

14.2.2 Fuzzy LP with Crisp Objective Function

A model in which the objective function is crisp—that is, has to be maximized or minimized—and in which the constraints are all or partially fuzzy is no longer symmetrical. The roles of objective functions and constraints are different; the latter define the decision space in a crisp or fuzzy way, and the former induce an order of the decision alternatives. Therefore the approach of models (14.3)–(14.5) is not applicable. The main problem is the scaling of the objective function (the domain of which is not normalized) when aggregating it with the (normalized) constraints. In very rare real cases, a scaling factor can be found that has a real justification.

The problem we face is the determination of an extremum of a crisp function over a fuzzy domain, which we have already discussed in section 7.2 of this book. In definition 7–3, we defined the notion of a maximizing set that we will specify here and use as a vehicle to solve our LP problem. Two approaches are conceivable:

1. The determination of the fuzzy set "decision."
2. The determination of a crisp "maximizing decision" by aggregating the objective function after appropriate transformations with the constraints.

1. The Determination of a Fuzzy Set "Decision." Orlovski [1977] suggests computing, for all α-level sets of the solution space, the corresponding optimal values of the objective function and considering as the fuzzy set "decision" the optimal values of the objective functions, with the degree of membership equal to the corresponding α-level of the solution space.

Definition 14–3 [Werners 1984]

Let $R_\alpha = \{x | x \in X, \mu_R(x) \geq \alpha\}$ be the α-level sets of the solution space and $N(\alpha) = \{x | x \in R_\alpha, f(x) = \sup_{x' \in R_\alpha} f(x')\}$ the set of optimal solutions for each α-level set.

The fuzzy set "*decision*" is then defined by the membership function

$$\mu_{\text{opt}}(x) = \begin{cases} \sup\limits_{x \in N(\alpha)} \alpha & \text{if } \; x \in \bigcup\limits_{\alpha > 0} N(\alpha) \\ 0 & \text{else} \end{cases}$$

The fuzzy set "*optimal values of the objective function*" has the membership function

$$\mu_f(r) = \begin{cases} \sup\limits_{x \in f^{-1}(r)} \mu_{\text{opt}}(x) & \text{if } \; r \in \mathbb{R}_1 \wedge f^{-1}(r) \neq \varnothing \\ 0 & \text{else} \end{cases}$$

$f(x)$ is the objective function with functional values r.

For the case of linear programming, the determination of the r's and $\mu_{\text{opt}}(x)$ can be obtained by parametric programming [Chanas 1983]. For each α, an LP of the following kind would have to be solved:

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{such that} && \alpha \leq \mu_i(x) \quad i = 1, \ldots, m \\ &&& x \in X \end{aligned} \tag{14.13}$$

The reader should realize, however, that the result is a fuzzy set and that the decision maker would have to decide which pair $(r, \mu_f f(r))$ he or she considers optimal if he or she wants to arrive at a crisp optimal solution.

Example 14–6 [Werners 1984]

Consider the LP-Model

$$\begin{aligned} \text{maximize} \quad & z = 2x_1 + x_2 \\ \text{such that} \quad & x_1 \leqq 3 \\ & x_1 + x_2 \leqq 4 \\ & 5x_1 + x_2 \leqq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The "tolerance intervals" of the constraints are $p_1 = 6$, $p_2 = 4$, $p_3 = 2$.

The parametric linear program for determining the relationships between $f(x) = r$ and degree of membership is then

$$\begin{aligned} \text{maximize} \quad & z = 2x_1 + x_2 \\ \text{such that} \quad & x_1 \leq 9 - 6\alpha \\ & x_1 + x_2 \leq 8 - 4\alpha \\ & 5x_1 + x_2 \leq 5 - 2\alpha \\ & x_1, x_2 \geq 0 \end{aligned}$$

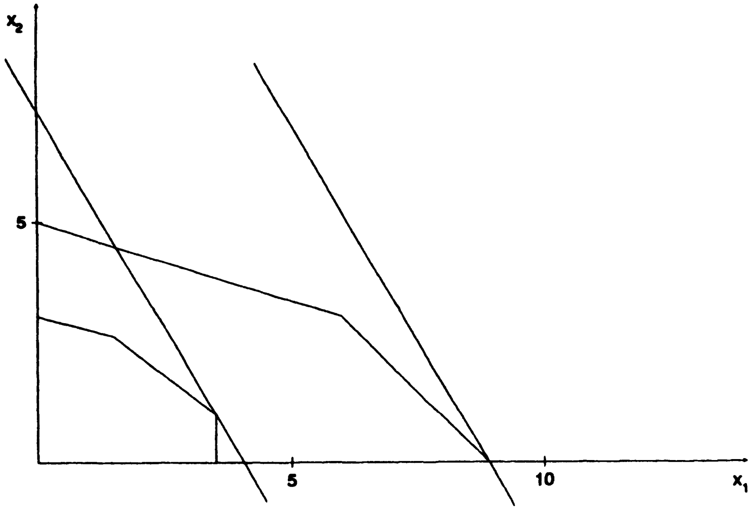


Figure 14–4. Feasible regions for $\mu_{\tilde{R}}(x) = 0$ and $\mu_{\tilde{R}}(x) = 1$.

Figure 14–4 shows the feasible regions for R_0 and R_1 for $\mu_{\tilde{R}}(x) = 0$ and $\mu_{\tilde{R}}(x) = 1$. Figure 14–5 shows the resulting membership function $\mu_f(r)$. Additionally, figure 14–5 shows the membership function of the goal and the fuzzy decision that will be discussed below.

Obviously, the decision maker has to decide which combination $(r\ \mu_f(r))$ he or she considers best.

Decision aids in this respect either can be derived from external sources or may depend on the problem itself. In the following, we shall consider an approach that suggests a crisp solution dependent on the solution space.

2. The Determination of a Crisp Maximizing Decision Some authors [Kickert 1978; Nguyen 1979; Zadeh 1972] suggest approaches based on the notion of a maximizing set, which seem to have some disadvantages (see Werners [1984]). We shall therefore present a model that is particularly suitable for the type of linear programming model we are considering here. Werners [1984] suggests the following definition.

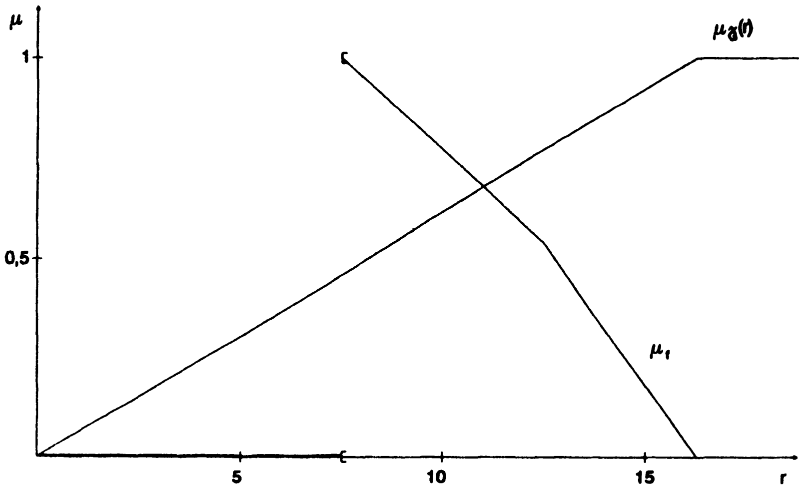


Figure 14–5. Fuzzy decision.

Definition 14–4

Let f: $X \rightarrow \mathbb{R}^1$ be the objective function, $\tilde{R}$ a fuzzy region (solution space), and $S(\tilde{R})$ the support of this region. The *maximizing set over the fuzzy region,* $\widetilde{MR}(f)$, is then defined by its membership function

$$\mu_{\widetilde{MR}(f)}(x) = \begin{cases} 0 & \text{if} \quad f(x) \le \inf\limits_{S(\tilde{R})} f \\ \dfrac{f(x) - \inf\limits_{S(\tilde{R})} f}{\sup\limits_{S(\tilde{R})} f - \inf\limits_{S(\tilde{R})} f} & \text{if} \quad \inf\limits_{S(\tilde{R})} f < f(x) < \sup\limits_{S(\tilde{R})} f \\ 1 & \text{if} \quad \sup\limits_{S(\tilde{R})} f \le f(x) \end{cases}$$

The intersection of this maximizing set with the fuzzy set "decision" (figure 14–5) could then be used to compute a maximizing decision x_0 as the solution with the highest a degree of membership in this fuzzy set. It does not seem reasonable that the judgment of the decision maker is calibrated by looking at the smallest value of f over the feasible region. A better benchmark would be the largest value for f that can be obtained at a degree of membership of 1 of the feasible region. This leads to the following definition.

Definition 14–5 [Werners 1984]

Let $f: X \to \mathbb{R}^1$ be the objective function, $\tilde{R}$ = fuzzy feasible region, $S(\tilde{R})$ = support of $\tilde{R}$, and R_1 = α-level cut of $\tilde{R}$ for $\alpha = 1$. The *membership function of the goal* (objective function) *given solution space* $\tilde{R}$ is then defined as

$$\mu_{\tilde{G}}(x) = \begin{cases} 0 & \text{if } f(x) \le \sup\limits_{R_1} f \\ \dfrac{f(x) - \sup\limits_{R_1} f}{\sup\limits_{S(\tilde{R})} f - \sup\limits_{R_1} f} & \text{if } \sup\limits_{R_1} f < f(x) < \sup\limits_{S(\tilde{R})} f \\ 1 & \text{if } \sup\limits_{S(\tilde{R})} f \le f(x) \end{cases}$$

The corresponding membership function in functional space is then

$$\mu_{\tilde{G}}(r) := \begin{cases} \sup\limits_{x \in f^{-1}(r)} \mu_{\tilde{G}}(x) & \text{if } r \in \mathbb{R}, f^{-1}(r) \neq \varnothing \\ 0 & \text{else} \end{cases}$$

Example 14–7

Consider the model of example 14–6. For this model, R_1 is the region defined by

$$\begin{aligned} x_1 &\le 3 \\ x_1 + x_2 &\le 4 \\ 5x_1 + x_2 &\le 3 \\ x &\ge 0 \end{aligned}$$

The supremum of f over this region is

$$\sup_{R_1} 2x_1 + x_2 = 7$$

Figure 14–5 shows the membership functions $\mu_f(r)$ and $\mu_{\tilde{G}}(r)$. Using the min-max approach, the resulting solution is $x_1^0 = 5.84$, $x_2^0 = .05$, $r_0 = 11.73$, and the attained degree of membership is $\mu_{\tilde{R}}(x_0) = .53$.

Let us now return to model (14.2) and modify it by considering the objective function to be crisp and by adding a set of crisp constraints $Dx \le b'$:

$$\begin{aligned} \text{maximize} \quad & f(x) = c^T x \\ \text{such that} \quad & \left.\begin{aligned} Ax &\lesssim b \\ Dx &\le b' \\ x &\ge 0 \end{aligned}\right\} \tilde{R} \end{aligned} \tag{14.14}$$

Let the membership functions of the fuzzy sets representing the fuzzy constraints be defined in analogy to equation (14.7) as

$$\mu_i(x)=\begin{cases}1 & \text{if} \quad A_i x \le b_i \\ \dfrac{b_i+p_i-A_i x}{p_i} & \text{if} \quad b_i < A_i x \le b_i + p_i \\ 0 & \text{if} \quad A_i x > b_i + p_i\end{cases} \tag{14.15}$$

The membership function of the objective function (14.5) can be determined by solving the following two LPs:

$$\begin{aligned} \text{maximize} \quad & f(x)=c^T x \\ \text{such that} \quad & Ax \le b \\ & Dx \le b' \\ & x \ge 0 \end{aligned} \tag{14.16}$$

yielding $\sup_{R_1} f = (c^T x)_{\text{opt}} = f_1$; and

$$\begin{aligned} \text{maximize} \quad & f(x)=c^T x \\ \text{such that} \quad & Ax \le b + p \\ & Dx \le b' \\ & x \ge 0 \end{aligned} \tag{14.17}$$

yielding $\sup_{S(\tilde{R})} f = (c^T x)_{\text{opt}} = f_0$.

The membership function of the objective function is therefore

$$\mu_{\tilde{G}}(x)=\begin{cases}1 & \text{if} \quad f_0 \le C^T x \\ \dfrac{c^T x - f_1}{f_0 - f_1} & \text{if} \quad f_1 < c^T x < f_0 \\ 0 & \text{if} \quad c^T x \le f_1\end{cases} \tag{14.18}$$

Now we have again achieved "symmetry" between constraints and the objective function, and we can employ the approach we used to derive model (14.9) as an equivalent formulation of model (14.2).
The equivalent model to (14.6) is

$$\begin{aligned} \text{maximize} \quad & \lambda \\ \text{such that} \quad & \lambda(f_0 - f_1) - c^T x \le - f_1 \\ & \lambda p + Ax \le b + p \\ & Dx \le b' \\ & \lambda \le 1 \\ & \lambda, x \ge 0 \end{aligned} \tag{14.19}$$

Example 14–8

We shall again consider the model in example 14–6. In example 14–7, we have computed $f_1 = 7$. By solving problem (14.17), we obtain $f_0 = 16$. Therefore problem (14.19) is

$$
\begin{aligned}
\text{maximize} \quad & \lambda \\
\text{such that} \quad & 9\lambda - 2x_1 - x_2 \leq -7 \\
& 6\lambda + x_1 \leq 9 \\
& 4\lambda + x_1 + x_2 \leq 8 \\
& 2\lambda + 5x_1 + x_2 \leq 5 \\
& \lambda \leq 1 \\
& \lambda, x_1, x_2 \geq 0
\end{aligned}
$$

The solution to this problem is $x_1^0 = 5.84$, $x_2^0 = 0$, $\lambda_0 = .52$.

Before we turn to fuzzy dynamic programming, it should be mentioned that on the basis of the approach described so far, suggestions have been published for a duality theory [Rödder and Zimmermann 1980], for sensitivity analysis in fuzzy linear programming [Hamacher, Leberling, and Zimmermann 1978], for integer fuzzy programming [Zimmermann and Pollatschek 1984], and for the use of other than linear membership functions and other operators [Werners 1988]. These topics will not, however, be discussed here. They have been discussed in more detail in Zimmermann [1987]. Other approaches introducing fuzziness into mathematical programming have been published by a number of authors (see, for instance, [Slowinski 1998], [Wang et al. 2001], [Sakawa and Nighizaki 2001], [Jamison and Lodwick 2001], [Sengupta et al. 2001]). Often these approaches have been developed in the context of multi objective decision making. In order to avoid duplication, these approaches will be mentioned at the end of the discussion of the vector-maximum problem in section 14–4.

14.3 Fuzzy Dynamic Programming

Traditional dynamic programming [Bellman 1957] is a technique well known in operations research and used to solve optimization problems that can be composed into subproblems of one variable (decision-variable) each. The idea underlying dynamic programming is to view the problem as a multistage decision process, the optimal policy to which can be determined recursively.

Generally the problem is formulated in terms of state variables, x_i; decision

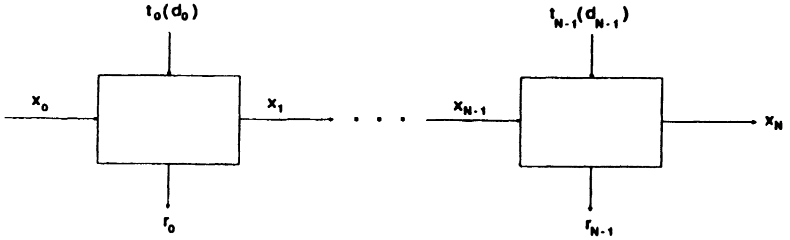


Figure 14–6. Basic structure of a dynamic programming model.

variables, d_i; stage rewards, $r_i\,(x_i, d_i)$; a reward function, $R_i(d_N, \ldots, d_{N-i}, x_N)$; and a transformation function, $t_i(d_i, x_i)$. Figure 14–6 illustrates the basic structure.

The problem is solved by solving recursively the following:

$$\max_{d_i} R_i(x_i, d_i) = \max_{d_i} r_i(x_i, d_i) \circ R_{i+1}(x_{i+1})$$

such that

$$x_{i+1} = t_i(x_i, d_i)$$
$$i = 1, \ldots, N-1$$

or

$$\max_{d_i} R_i(x_i, d_i) = \max_{d_i} \{r_i(x_i, d_i) \circ R_{i+1}(t_i(x_i, d_i))\}$$

All variables, rewards, and transformations are supposed to be crisp.

14.3.1 Fuzzy Dynamic Programming with Crisp State Transformation Function

In their famous paper, Bellman and Zadeh [1970] suggested for the first time a fuzzy approach to this type of problem. Conceivably they based their considerations on the symmetrical model of a decision as defined in definitions 14–1 and 14–2. The following terms will be used to define the fuzzy dynamic programming model [Bellman and Zadeh 1970, B-151]: $\tilde{X}_i \in \tilde{X}$, $i = 0, \ldots,$ N: (crisp) state variable where $\tilde{X} = \{\tau_1, \ldots, \tau_N\}$ is the set of values permitted for the state variables; $d_i \in \tilde{D}$, $i = 1, \ldots, N$: (crisp) decision variable where $\tilde{D} = \{\alpha_1, \ldots, \alpha_m\}$ is the set of possible decisions.

$$x_{i+1} = t(x_i, d_i)\text{: (crisp) transformation function}$$

For each stage t, $t = 0, \ldots, N-1$, we define:

1. a fuzzy constraint $\tilde{C}_t$ limiting the decision space and characterized by its membership function

$$\mu_{\tilde{C}_t}(d_t)$$

2. a fuzzy goal $\tilde{G}_N$ characterized by the membership function

$$\mu_{\tilde{G}_N}(x_N)$$

The problem is to determine the maximizing decision

$$\tilde{D}^0 = \{d_i^0\} \quad i = 0, \ldots, N, \quad \text{for a given } x_0$$

The Model. According to definition 14–1, the fuzzy set decision is the "confluence" of the constraints and the goal(s), that is,

$$\tilde{D} = \bigcap_{t=0}^{N-1} \tilde{C}_t \cap \tilde{G}_N$$

Using the min-operator for the aggregation of the fuzzy constraints and the goal, the membership function of the fuzzy set decision is

$$\mu_{\tilde{D}}(d_0, \ldots, d_{N-1}) = \min\{\mu_{\tilde{C}_0}(d_0), \ldots, \mu_{\tilde{C}_{N-1}}(d_{N-1}), \mu_{\tilde{G}_N}(x_N)\} \tag{14.20}$$

The membership function of the maximizing decision is then

$$\mu_{\tilde{D}^0}(d_0^0, \ldots, d_{N-1}^0) = \max_{d_0, \ldots, d_{N-2}} \max_{d_{N-1}} [\min\{\mu_{\tilde{C}_0}(d_0), \ldots, \mu_{\tilde{G}_N}(t_N(x_{N-1}, d_{N-1}))\}] \tag{14.21}$$

where d_i^0 denotes the optimal decision on stage i. If K is a constant and g is any function of d_{N-1}, we can write

$$\max_{d_{N-1}} \min\{g(d_{N-1}), K\} = \min\{K, \max_{d_{N-1}} g(d_{N-1})\}$$

and equation (14.21) can be expressed as

$$\mu_{\tilde{D}^0}(d_0^0, \ldots, d_{N-1}^0) = \max_{d_0, \ldots, d_{N-1}} \min\{\mu_{\tilde{C}_0}(d_0), \ldots, \mu_{\tilde{G}_{N-1}}(x_{N-1})\} \tag{14.22}$$

with

$$\mu_{\tilde{G}_{N-1}}(x_{N-1}) = \max_{d_{N-1}} \min\{\mu_{\tilde{C}_{N-1}}(d_{N-1}), \mu_{\tilde{G}_N}(t_N(x_{N-1}, d_{N-1}))\} \tag{14.23}$$

We can thus determine $\tilde{D}^0$ recursively.

Example 14–9 [Bellman and Zadeh 1970, B-153]

Let $\tilde{d}_1$, $\tilde{d}_2$ be the two decision variables, the possible values of which can be α_1, α_2. The state variables are x_t, $t = 0, \ldots, 2$ with a finite range $X = \{\tau_1, \tau_2, \tau_3\}$.

The fuzzy constraints for $t = 0$ and $t = 1$ are

$$\tilde{C}_0(\alpha_i) = \{(\alpha_1, .7), (\alpha_2, 1)\}$$
$$\tilde{C}_1(\alpha_i) = \{(\alpha_1, 1), (\alpha_2, .6)\}$$

The fuzzy goal is specified as

$$\tilde{G}(x_2) = \{(\tau_1, .3), (\tau_2, 1), (\tau_3, .8)\}$$

and the crisp transformation function is defined by the following matrix:

	x_t		
d_t	τ_1	τ_2	τ_3
α_1	τ_1	τ_3	τ_1
α_2	τ_2	τ_1	τ_3

Solution. Using equation (14.23) we can compute the fuzzy goal induced at $t = 1$ as follows: We start at stage $t = 2$. The state-decision combinations that yield τ_i on state $t = 1$ are obtained from the above matrix.

So we can compute:

$$\begin{aligned}
\mu_{\tilde{G}_1}(\tau_1) &= \max_{d_1} \{\min [\mu_{\tilde{C}_1}(d_1), \mu_{\tilde{G}_2}(t(\tau_1, \alpha_1))] \\
&\qquad\quad \min [\mu_{\tilde{C}_1}(d_1), \mu_{\tilde{G}_2}(t(\tau_1, \alpha_2))]\} \\
&= \max \{\min [1, .3], \min [.6, 1]\} \\
&= \max \{.3, .6\} = .6 \\
&\rightarrow d_1^0 = \alpha_2
\end{aligned}$$

$$\begin{aligned}
\mu_{\tilde{G}_1}(\tau_2) &= \max \{\min [1, .8], \min [.6, .3]\} \\
&= \max \{.8, .3\} = .8 \\
&\rightarrow d_1^0 = \alpha_1
\end{aligned}$$

$$\begin{aligned}
\mu_{\tilde{G}_1}(\tau_3) &= \max \{\min [1, .3], \min [.6, .8]\} \\
&= \max \{.3, .6\} = .6 \\
&\rightarrow d_1^0 = \alpha_2
\end{aligned}$$

$$\begin{aligned}
\mu_{\tilde{G}_0}(\tau_1) &= \max \{\min [.7, .6], \min [1, .8]\} \\
&= .8 \\
&\rightarrow d_0^0 = \alpha_2
\end{aligned}$$

$$\begin{aligned}\mu_{\tilde{G}_0}(\tau_2) &= \max\{\min[.7, .6], \min[1, .6]\}\\ &= .6\\ &\rightarrow d_0^0 = \alpha_2 \text{ or } \alpha_2\\ \mu_{\tilde{G}_0}(\tau_3) &= \max\{\min[.7, .6], \min[1, .6]\}\\ &= .6\\ &\rightarrow d_0^0 = \alpha_1 \text{ or } \alpha_2\end{aligned}$$

Thus for

$$\begin{aligned}x_0 = \tau_1 &: d_0^0 = \alpha_2, \quad d_1^0 = \alpha_1\\ &\quad \text{with } \mu_{\tilde{G}_2^0} = .8\\ x_0 = \tau_2 &: d_0^0 = \alpha_1, \quad d_1^0 = \alpha_2 \quad \text{or}\\ &\quad d_0^0 = \alpha_2, \quad d_1^0 = \alpha_2\\ &\quad \text{with } \mu_{\tilde{G}_2^0} = .6\\ x_0 = \tau_3 &: d_0^0 = \alpha_1, \quad d_1^0 = \alpha_2 \quad \text{or}\\ &\quad d_0^0 = \alpha_2, \quad d_1^0 = \alpha_2\\ &\quad \text{both with } \mu_{\tilde{G}_2^0} = .6\end{aligned}$$

14.4 Fuzzy Multicriteria Analysis

In the recent past, it has become more and more obvious that comparing the desirability of different means of action, judging the suitability of products, or determining "optimal" solutions in decision problems cannot be done in many cases by using a single criterion or a single objective function. This area, multicriteria decision making, has led to numerous evaluation schemes (e.g., in the areas of cost–benefit analysis and marketing) and to the formulation of vector-maximum problems in mathematical programming.

Two major areas have evolved, both of which concentrate on decision making with several criteria: Multi Objective Decision Making (MODM) and Multi Attribute Decision Making (MADM). The main difference between these two directions is that the former concentrates on continuous decision spaces, primarily on mathematical programming with several objective functions, and the latter focuses on problems with discrete decision spaces. There are some exceptions to this rule (e.g., integer programming with multiple objectives), but for our purposes this distinction seems to be appropriate.

The literature on multicriteria decision making has grown tremendously in the recent past. We shall only mention one survey reference for each of these two areas: Hwang and Yoon [1981] for MADM and Hwang and Masud [1979] for MODM. Fuzzy set theory has contributed to MODM as well as to MADM. We shall illustrate these contributions by describing one model in each of these areas. This topic has been treated in much more detail in the volume on fuzzy sets and decision analysis [Zimmermann 1987].

14.4.1 Multi Objective Decision Making (MODM)

In mathematical programming, the MODM problem is often called the "vector-maximum" problem, and was first mentioned by Kuhn and Tucker [1951].

Definition 14–6

The *vector-maximum problem* is defined as

$$\text{"maximize"} \{Z(x) \mid x \in X\}$$

where $Z(x) = (z_1(x), \ldots, z_k(x))$ is a vector-valued function of $x \in \mathbb{R}^n$ into $\mathbb{R}^k$ and X is the "solution space."
Two stages can generally be distinguished, at least categorically, in vector-maximum optimization:

1. the determination of efficient solutions, and
2. the determination of an optimal compromise solution.

Definition 14–7

Let "max" $\{Z(x) \mid x \in X\}$ be a vector-maximum problem as defined in definition 14–6. $\bar{x}$ is an *efficient solution* if there is no $\hat{x} \in X$ such that

$$z_i(\hat{x}) \geq z_i(\bar{x}) \quad i = 1, \ldots, k$$

and

$$z_i(\hat{x}) > z_i(\bar{x}) \quad \text{for at least one } i = 1, \ldots, k$$

The set of all efficient solutions is generally called the "*complete solution.*"

Definition 14–8

An *optimal compromise solution* of a vector-maximum problem is a solution $x \in X$ that is preferred by the decision maker to all other solutions, taking into consideration all criteria contained in the vector-valued objective function. It is generally accepted that an optimal compromise solution has to be an efficient solution according to definition 14–7.

In the following, we shall restrict our considerations to the determination of optimal compromise solutions in linear programming problems with vector-valued objective functions.

Three major approaches are known to single out one specific solution from the set of efficient solutions which qualifies as an "optimal" compromise solution:

1. the utility approach [see, e.g., Keeney and Raiffa 1976],
2. goal programming [see, e.g., Charnes and Cooper 1961], and
3. interactive approaches [see, e.g., Dyer 1973]

The first two of these approaches assume that the decision maker can specify his or her "preference function" with respect to the combination of the individual objective functions in advance, either as "weights" (utilities) or as "distance functions" (concerning the distance from an "ideal solution," for example). Generally these two approaches assume that the combination of the individual objective functions that arrives at the compromise solution with the highest overall utility is achieved by linear combinations (i.e., adding the weighted individual objective functions). The third approach uses only local information in order to arrive at an acceptable compromise solution.

The following example illustrates a fuzzy approach to this problem.

Example 14–10

A company manufactures two products 1 and 2 on given capacities. Product 1 yields a profit of \$2 per piece and product 2 of \$1 per piece. Product 2 can be exported, yielding a revenue of \$2 per piece in foreign countries; product 1 needs imported raw materials of \$1 per piece. Two goals are established: (1) profit maximization and (2) maximum improvement of the balance of trade, that is, maximum difference of exports minus imports. This problem can be modeled as follows:

$$\text{"maximize" } Z(x) = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{matrix} \text{(effect on balance of trade)} \\ \text{(profit)} \end{matrix}$$

such that

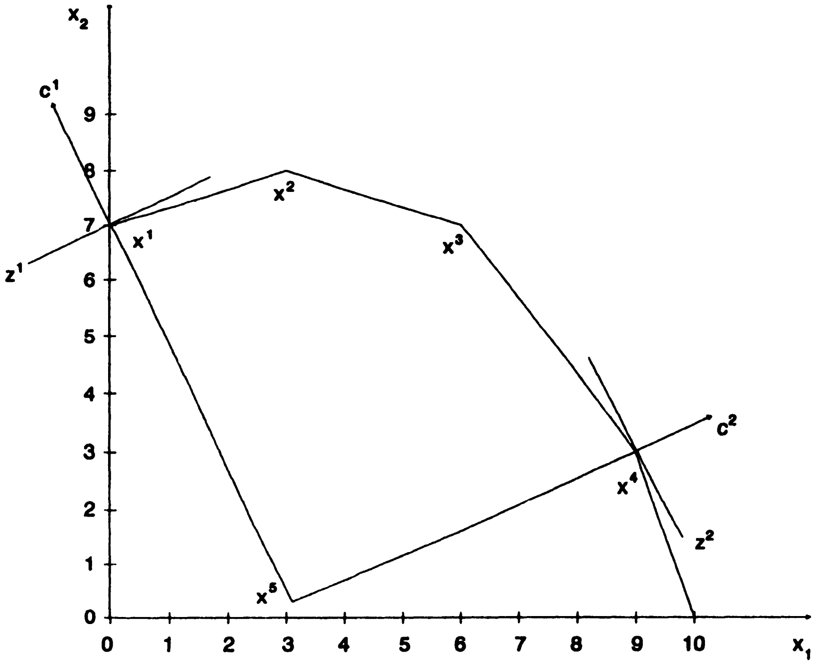


Figure 14–7. The vector-maximum problem.

$$-x_1 + 3x_2 \leq 21$$
$$x_1 + 3x_2 \leq 27$$
$$4x_1 + 3x_2 \leq 45$$
$$3x_1 + x_2 \leq 30$$
$$x_1, x_2 \geq 0$$

Figure 14–7 shows the solution space of this problem. The "complete solution" is the edge $x^1 - x^2 - x^3 - x^4$. x^1 is optimal with respect to objective function $z_1(x) = -x_2 + 2x_2$ (i.e., best improvement of balance of trade). x^4 is optimal with respect to objective function $z_2(x) = 2x_1 + x_2$ (profit). The "optimal" values are $z_1(x^1) = 14$ (maximum net export) and $z_2\,(x^4) = 21$ (maximum profit), respectively. For $x^1 = (7;\, 0)^T$, total profit is $z_2(x^1) = 7$ and $x^4 = (9;\, 3)^T$ yields $z_1(x^4) = -3$, that is, a net import of 3. Solution $x^5 = (3.4;\, 0.2)^T$ is the solution that yields $z_1(x^5) = -3$, $z_2(x^5) = 7$, which is the lowest "justifiable" value of the objective functions in the sense that a further decrease of the value of one objective function cannot be balanced or even counteracted by an increase in the value of the other objective function.

To solve problems of the kind shown in example 14–10, we can use the following approach. We first assume that either the decision maker can specify aspiration levels for the objective functions or we define properties of the solution space for "calibration" of the objective functions. Let us consider the objective functions as fuzzy sets of the type "solutions acceptable with respect to objective function 1." In example 14–10, we would have to construct two fuzzy sets: "Solutions acceptable with respect to objective function 1" and "solutions acceptable with respect to objective function 2." As calibration points, we shall use the respective "individual optima" and the "least justifiable solution."

The membership functions $\mu_1(x)$ and $\mu_2(x)$ of the fuzzy sets characterizing the objective functions rise linearly from 0 to 1 at the highest achievable values of $z_1(x) = 14$ and $z_2(x) = 21$, respectively.

This means that we assume that the level of satisfaction with respect to the improvement of the balance of trade rises from 0 for imports of 3 units or more to 1 for exports of 14 and more; and the satisfaction level rises with respect to profit from 0 if the profit is 7 or less to 1 if total profit is 21 or more.

$$\mu_1(x) = \begin{cases} 0 & \text{for} \quad z_1(x) \le -3 \\ \dfrac{z_1(x)+3}{17} & \text{for} \quad -3 < z_1(x) \le 14 \\ 1 & \text{for} \quad 14 < z_1(x) \end{cases}$$

$$\mu_2(x) = \begin{cases} 0 & \text{for} \quad z_2(x) \le 7 \\ \dfrac{z_2(x)-7}{14} & \text{for} \quad 7 < z_2(x) \le 21 \\ 1 & \text{for} \quad 21 < z_2(x) \end{cases}$$

We are now faced with a problem of type (14.3) in which crisp constraints have been added (i.e., the problem consists of two rows representing our fuzzified objectives and four crisp constraints). We can now employ formulation (14.12).

Example 14–10 (continuation)

In analogy to formulation (14.12) and including the crisp constraints, we arrive at the following problem formulation:

$$\begin{aligned}
\text{maximize} \quad & \lambda \\
\text{such that} \quad & \lambda \leq -0.05882x_1 + 0.117x_2 + 0.1764 \\
& \lambda \leq +0.1429\,x_1 + 0.714x_2 - 0.5 \\
& 21 \geq -x_1 + 3x_2, \\
& 27 \geq x_1 + 3x_2, \\
& 45 \geq 4x_1 + 3x_2, \\
& 30 \geq 3x_1 + x_2, \\
& x \geq 0,
\end{aligned}$$

depicted in figure 14.8.

The maximum degree of "overall satisfaction" ($\lambda_{max} = 0.74$) is achieved for the solution $x_0 = (5.03; 7.32)^T$. This is the "maximizing solution," which in our example yields a profit of \$17.38 and an export contribution of \$4.58. The basic solution x^1 and x^4 yield $\lambda = 0$.

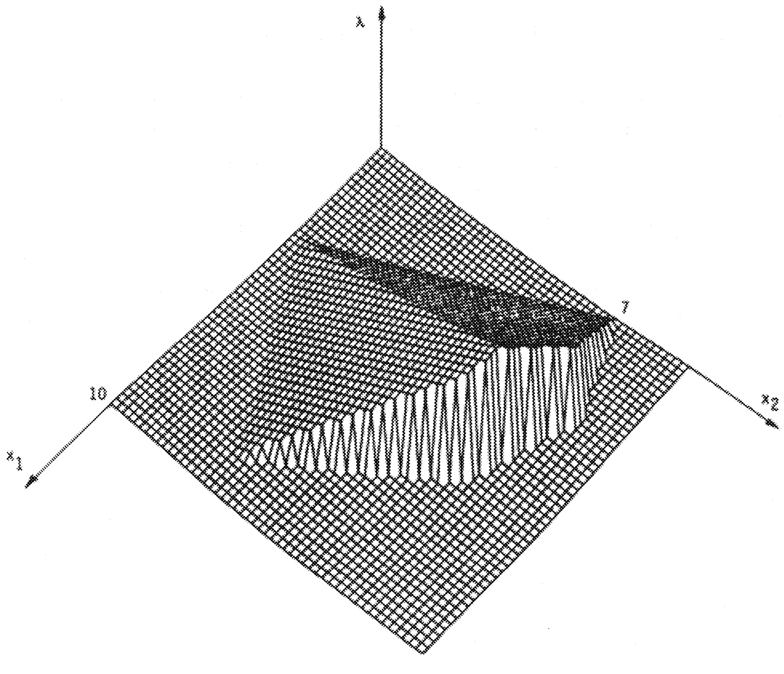


Figure 14–8. Fuzzy LP with min-operator.

In contrast to the usual vector-maximum models, the efficient solutions contained in the "complete solution" are ordered (distinguishable) by their degree of membership to the fuzzy set decision. It should be obvious that the approach described above can only be applied if the "symmetrical model" of a decision (definition 14–1) is accepted. Otherwise, we will have to use approaches applicable to problem (14.13), these, however, will not be discussed in this volume.

At the beginning of section 14.2, many simplifying assumptions were pointed out that are generally accepted in traditional linear programming models. These assumptions concerned the use of real numbers rather than fuzzy numbers for the coefficients of linear programming as well as the use of crisp relations rather than fuzzy. One approach used in section 14.2 for the fuzzification of crisp mathematical programming problems seems to be computationally very efficient, well applicable in practice, and understandable by practioners. In the literature, the reader will find numerous different approaches that, from a mathematical point of view, are quite interesting. It would certainly exceed the scope of this book to describe the majority of these suggestions. We shall, however, mention a few of them. The reader will find quite a number of references to other approaches in the bibliography at the end of the book.

Approaches that use fuzzy sets to describe the parameter of linear programming models can be traced, in particular, to the paper by Negoita, Minouiu, and Stan [1976]. They use fuzzy sets to describe the parameters of the matrix A and the capacity vector b and then formulate for each α the respective α-cuts. The resulting crisp problem can then be solved by the usual LP codes. If the membership functions have only a finite number of values, an optimal alternative and an objective function value can be determined for each case. This approach, however, is connected with a high computational effort. Afterwards the decision maker has to choose a desirable degree of membership and the associated solution. Kacprzyk and Orlovski [1987], in their review article, mention a number of additional references in which special representations of fuzzy parameters are used. Here we shall mention only the work of Tanaka and Asai [1984], who use triangular membership functions, and Ramik and Rimanek [1985, 1989], who use fuzzy parameters in LR representation and replace each resulting fuzzy relation by four strict relations.

Other authors consider nonlinear vector-maximum problems in which all parameters are defined fuzzily. Sakawa and Yano [1987], for instance, formulate a fuzzy nonlinear vector-maximum problem with fuzzy parameters $\tilde{a}_e$, L = 1, . . . , k in the k objective functions and $\tilde{b}_i$, $i = 1, \ldots, m$ in the m constraints. Here the fuzzy parameters are regarded as real-valued fuzzy numbers. For each α-degree, a crisp equivalent model can be formulated for which the values of the fuzzy numbers can be considered as variables subject to the condition that they belong to the fuzzy number at least with the degree of membership α. Sakawa and Jano [1987] define the notion of an α-pareto-optimal solution in generalizing the classical pareto-optimality with respect to the crisp equivalent models. The authors suggest an interactive algorithm that leads the decision maker to a satisfying solution. The decision maker has to provide as starting values the desired α and the aspiration level for the objective function. The algorithm then solves an equivalent model that minimizes for a given α the deviation from the aspiration level and supplies additional trade-off information to the decision maker. This approach assumes that the decision maker can choose the states that are expressed in the fuzzy numbers. Therefore, this approach seems to be only suitable if the decision maker can really influence these values, that is, if they are not dependent on the environment. Because it is assumed that the parameters are variables, the resulting α-model is at least quadratic, even if the basic model is linear.

If the fuzzy coefficients are the result of insufficient information that can be improved by additional effort, an optimal context-dependent allocation of additional effort is of interest. Tanaka, Ishihashi, and Asai [1986] discuss the value of additional information and suggest a model for the allocation of information on the basis of sensitivity analysis. In the recent past, fuzzy models have also been suggested for fractional programming, integer programming, geometric programming, and other versions of mathematical programming problems. Of particular interest is the application of possibility theory to mathematical programming suggested by Buckley [1988a, 1988b], and the papers by Arikan and Güngör [2001], Abd El-Wahed and Abo-Sinna [2001], and Jamison and Lodwich [2001].

14.4.2 Multi Attributive Decision Making (MADM)

The general multi attributive decision-making model can be defined as follows.

Definition 14–9

Let $X = \{x_i \mid i = 1, \ldots, n\}$ be a (finite) set of decision alternatives and $G = \{g_j \mid j = 1, \ldots, m\}$ a (finite) set of goals according to which the desirability of an action is judged. Determine the optimal alternative x^0 with the highest degree of desirability with respect to all relevant goals g_j.

Most approaches in MADM consist of two stages:

1. the aggregation of the judgments with respect to all goals and per decision alternative, and
2. the rank ordering of the decision alternatives according to the aggregated judgments.

In crisp MADM models, it is usually assumed that the final judgments of the alternatives are expressed as real numbers. In this case, the second stage does not pose any particular problems and suggested algorithms concentrate on the first stage. Fuzzy models are sometimes justified by the argument that the goals, g_j, or their attainment by the alternatives, x_i, respectively, cannot be defined or judged crisply but only as fuzzy sets. In this case, the final judgments are also represented by fuzzy sets, which have to be ordered to determine the optimal alternative. Then the second stage is, of course, by far not trivial.

In the following, we shall describe two fuzzy MADM models—the first one, by Yager, because it shows very clearly the general structure of the problem and the second, by Baas and Kwakernaak, because many of the publications refer to this model, which is one of the first of this kind published.

Model 14–1! [Yager 1987]

Let $X = \{x_1, \ldots, x_n\}$ be a set of alternatives. The goals are represented by the fuzzy sets $\tilde{G}_j$, $j = 1, \ldots, m$. The "importance" (weight) of goal j is expressed by w_j, the "attainment" of goal $\tilde{G}_j$ by alternative x_i is expressed by the degree of membership $\mu_{\tilde{G}_{2j}}(x_i)$.

The decision is defined in line with definition 14–1 as the intersection of all fuzzy goals, that is,

$$\tilde{D} = \tilde{G}_1^{w_1} \cap \tilde{G}_2^{w_2} \cap \ldots \cap \tilde{G}_m^{w_m}$$

and the optimal alternative is defined as that achieving the highest degree of membership in $\tilde{D}$.

The rationale behind using the weights as exponents to express the importance of a goal can be found in definition 9–3: There the modifier "very" was defined as the squaring operation. Thus the higher the importance of a goal, the larger should be the exponent of its representing fuzzy set, at least for normalized fuzzy sets and when using the min-operator for the intersection of the fuzzy goals. Yager concentrates on the problem of determining the weights of the goals. As a solution to that problem, he suggests Saaty's hierarchical procedure for determining weights by computing the eigenvectors of the matrix M of relative weights of subjective estimates [Saaty 1978]:

> The membership grade in all objectives having little importance ($w < 1$) becomes larger, and while those in objectives having more importance ($w > 1$) become smaller. This has the effect of making the membership function of the decision subset D, which is the min value of each X over all objectives, being more determined by the important objectives, which is as it should be. Furthermore, this operation (min) makes particu-

larly small those alternatives that are bad in important objectives, therefore when we select the x_i that maximizes D, we will be very unlikely to pick one of these [Yager 1978, p. 90].

The solution procedure can now be described as follows: Given the set $X = \{x_1, \ldots, x_n\}$ and the degrees of membership $\mu_{\tilde{G}_j}(x_i)$ of all x_i in the fuzzy sets $\tilde{G}_j$ representing the goals,

1. Establish by pairwise comparison the relative importance, α_i, of the goals among themselves. Arrange the α_i in a matrix M.

$$M = \begin{bmatrix} \frac{\alpha_1}{\alpha_1} & \frac{\alpha_1}{\alpha_2} & \cdots\cdots & \frac{\alpha_1}{\alpha_n} \\ \frac{\alpha_2}{\alpha_1} & & & \vdots \\ \frac{\alpha_n}{\alpha_1} & & & \frac{\alpha_n}{\alpha_n} \end{bmatrix}$$

2. Determine consistent weights w_j for each goal by employing Saaty's eigenvector method.
3. Weight the degrees of goal attainment, $\mu_{\tilde{G}_j}(x_i)$ exponentially by the respective w_j. The resulting fuzzy sets are $(\tilde{G}_j(x_i))^{w_j}$
4. Determine the intersection of all $(\tilde{G}_j(x_i))^{w_j}$:

$$\tilde{D} = \left\{\left(x_i, \min_j (\mu_{\tilde{G}_j}(x_i)^{w_j}) \mid i = 1, \ldots, n; j = 1, \ldots, m\right)\right\}$$

5. Select the x_i with largest degree of membership in $\tilde{D}$ as the optimal alternative.

Example 14–11 [Yager 1978, p. 94]

Let $X = \{x_1, x_2, x_3\}$, and let the goals be given as

$$\tilde{G}_1(x_i) = \{(x_1, .7), (x_2, .5), (x_3, .4)\}$$
$$\tilde{G}_2(x_i) = \{(x_1, .3), (x_2, .8), (x_3, .6)\}$$
$$\tilde{G}_3(x_i) = \{(x_1, .2), (x_2, .3), (x_3, .8)\}$$
$$\tilde{G}_4(x_i) = \{(x_1, .5), (x_2, .1), (x_3, .2)\}$$

The subjective evaluations have resulted in the following matrix of weights:

$$M = \begin{array}{c c} & \begin{array}{cccc} \tilde{G}_1 & \tilde{G}_2 & \tilde{G}_3 & \tilde{G}_4 \end{array} \\ \begin{array}{c} \tilde{G}_1 \\ \tilde{G}_2 \\ \tilde{G}_3 \\ \tilde{G}_4 \end{array} & \begin{bmatrix} 1 & 3 & 7 & 9 \\ \frac{1}{3} & 1 & 6 & 7 \\ \frac{1}{7} & \frac{1}{6} & 1 & 3 \\ \frac{1}{9} & \frac{1}{7} & \frac{1}{3} & 1 \end{bmatrix} \end{array}$$

Via Saaty's method, we obtain the vector

$$w = \{w_1, w_2, w_3, w_4\} \quad \text{as}$$
$$\mathrm{w} = \{2.32, 1.2, .32, .16\}$$

Exponential weighting of $\tilde{G}_j(x_i)$ by their respective weight yields

$$\tilde{G}_1(x_i)^{2.32} = \{(x_1, .44), (x_2, .2), (x_3, .12)\}$$
$$\tilde{G}_2(x_i)^{1.2} = \{(x_1, .24), (x_2, .76), (x_3, .54)\}$$
$$\tilde{G}_3(x_i)^{.32} = \{(x_1, .6), (x_2, .68), (x_3, .93)\}$$
$$\tilde{G}_4(x_i)^{.16} = \{(x_1, .9), (x_2, .69), (x_3, .77)\}$$

The fuzzy set decision $\tilde{D}$, as the intersection of the $\tilde{G}^{\alpha_j}{}_j(x_i)$, becomes

$$\tilde{D} = \{(x_1, .24), (x_2, .2), (x_3, .12)\}$$

and the optimal alternative is x_1 with a degree of membership in $\tilde{G}$ of $\mu_{\tilde{D}}(x_1)$ = .24.

Model 14–2 [Baas and Kwakernaak 1977]

Let again $X = \{x_i \mid i = 1, \ldots, n\}$ be the set of alternatives and $G = \{g_j \mid j = 1, \ldots, m\}$ the set of goals. r_{ij} is the "rating" of alternative i with respect to goal j, and $w_j \in \mathbb{R}^1$ is the weight (importance) of goal j. It is assumed that the rating of alternative i with respect to goal j is fuzzy and is represented by the membership function $\mu_{\tilde{R}_{ij}}(r_{ij})$ on $\mathbb{R}^1$.

Similarly, the weight (relative importance) of goal j is represented by a fuzzy set w_j with membership function $\mu_{wj}(w_j)$. All fuzzy sets are assumed to be normalized (i.e., have finite supports and take on the value 1 at least once!).

Step 1. The evaluation of an alternative x_i is , by contrast to model 14.1, assumed to be a fuzzy set that is computed on the basis of the r_{ij} and w_j as follows: Consider a function g: $\mathbb{R}^{2m} \to \mathbb{R}$ defined by

$$g(z) = \frac{\sum_{j=1}^{m} w_j r_j}{\sum_{j=1}^{m} w_j} \tag{14.24}$$

with $z = (w_1, \ldots, w_m, r_1, \ldots, r_m)$.

On the product space $\mathbb{R}^{2n}$, a membership function μ_{zi} is defined as

$$\mu_{zi}(z) = \min\{\min_{j=1,\ldots,m}(\mu_{w_j}(w_j), \min_{k=1,\ldots,m}(\mu_{\tilde{R}_{ik}}(r_k))\} \tag{14.25}$$

Through the function g, the fuzzy set $\tilde{Z} = (\mathbb{R}^{2m}, \mu_{zi})$ induces a fuzzy set $\tilde{R}_i = (\mathbb{R}, \mu_{\tilde{R}_i})$ with the membership function

$$\mu_{\tilde{R}_i}(\bar{r}) = \sup_{z:g(z)=\bar{r}} \mu_{z_i}(z) \quad \bar{r} \in \mathbb{R} \tag{14.26}$$

$\mu_{\tilde{R}_i}(\bar{r})$ is the final rating of alternative x_i on the basis of which the "rank ordering" is performed in step 2.

Step 2. For the final ranking of the x_i, Baas and Kwakernaak start from the observation that if the x_i had received crisp rating r_i then a reasonable procedure would select the x_i that have received the highest rating, that is, would determine the set of preferred alternatives as $\{i \in I \mid r_i \geq r_j, \forall j \in I\}, I = \{1, \ldots, n\}$.

Since here the final ratings are fuzzy, the problem is somewhat more complicated. The authors suggest in their model two different fuzzy sets in addition to $\tilde{R}_i$, which supply different kinds of information about the preferability of an alternative.

a. They first determine the conditional set $(I \mid \tilde{R})$ with the characteristic function

$$\mu_{(I|\tilde{R})}(i \mid \bar{r}_1, \ldots, \bar{r}_n) = \begin{cases} 1 & \text{if} \quad \bar{r}_i \geq \bar{r}_j \quad \forall j \in I \\ 0 & \text{else} \end{cases} \tag{14.27}$$

This "membership function" expresses that a given alternative x_i belongs to the preferred set iff

$$\bar{r}_i \geq \bar{r}_j \quad \forall j \in I$$

The final fuzzy ratings $\tilde{R}$ define on $\mathbb{R}^n$ a fuzzy set $\tilde{R} = (\mathbb{R}^n, \mu_{\tilde{R}})$ with the membership function

$$\mu_{\tilde{R}}(\bar{r}_1, \ldots, \bar{r}_n) = \min_{i=1,\ldots,n} \mu_{\tilde{R}_i}(\bar{r}_i) \tag{14.28}$$

This fuzzy set together with the conditional fuzzy set (14.27) induces a fuzzy set $\tilde{I} = (I, \mu_{\tilde{I}})$ with the membership function

$$\mu_{\tilde{I}}(i) = \sup_{\bar{r}_1,\ldots,\bar{r}_n} (\min\{\mu_{(\tilde{I}|\tilde{R})}(i \mid \bar{r}_1,\ldots,\bar{r}_n), \mu_{\tilde{R}}(\bar{r}_1,\ldots,\bar{r}_n)\}) \tag{14.29}$$

which can be interpreted as the degree to which alternative x_i is the best alternative. If there is a unique i, then x_i corresponds to the alternative that maximizes equation (14.29) if the w_j and r_{ij} are set to the values at which $\mu_{\tilde{w}_j}(w_j)$ and $\mu_{\tilde{R}_{ij}}(r_{ij})$, respectively, attain their supremum, namely 1.

b. This is, of course, not all the information that can be provided. x_i might not be the unique best alternative, but there might be some x_i attaining their maximum degree of membership at r^*. They might, however, be represented by different fuzzy sets $\tilde{r}_{ij}$.

Baas and Kwakernaak therefore try to establish another criterion that might be able to distinguish such "preferable" alternatives from each other and rank them:

If the final ratings are crisp, $\tilde{r}_1, \ldots, \tilde{r}_n$, then

$$p_i = \bar{r}_i - \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{r}_j$$

for fixed i, can be used as a measure of preferability of alternative x_i over all others.

If the ratings $\tilde{r}_i$ are fuzzy, then the mapping h_i: $\mathbb{R}^m \to \mathbb{R}$ induces a fuzzy set $\tilde{P}_i = (\mathbb{R}, \mu_{\tilde{P}_i})$ with the membership function

$$\mu_{\tilde{P}_i}(p) = \sup_{h_i(\bar{r}_1,\ldots,\bar{r}_n)=p} \mu_{\tilde{R}}(\bar{r}_1,\ldots,\bar{r}_n) \tag{14.30}$$

in which $\mu_{\tilde{R}}$ is defined by equation (14.28).

This fuzzy set can be used to judge the degree of preferability x_i over all other alternatives.

The computational aspects for determining all the fuzzy sets mentioned above shall not be discussed here; models 1 and 2 have been described because of their illustrative value. Baas and Kwakernaak mention and prove special conditions for the membership functions to make computations possible.

To summarize: Three kinds of informations are provided:

1. $\mu_{\tilde{R}_i}(\bar{r})$ as the fuzzy rating of xi.
2. $\mu_{\tilde{I}}(i)$ as the degree to which x_i is best alternative, and
3. $\mu_{\tilde{P}_i}(p)$ as the degree of preferability of x_i over all other alternatives.

Example 14–12 [Baas and Kwakernaak 1977, p. 54]

Let $X = \{x_1, x_2, x_3\}$ be the set of available alternatives and $G = \{g_1, g_2, g_3, g_4\}$ the set of goals. The weights and the ratings of the alternatives with respect to the goals are given as normalized fuzzy sets that resemble the terms of a linguistic variable (see definition 9–1). Figure 14–9 depicts the fuzzy sets representing weights and ratings. Table 14-1 gives the assumed ratings for all alternatives and goals and the respective weights. Figure 14-10 shows the $\mu_{\tilde{R}_i}(r_i)$ (final ratings for alternatives x_1, x_2, x_3).

The degrees of membership of the alternatives to the fuzzy set $(I, \mu_{\tilde{I}})$, that is, the degrees to which alternatives x_i are best, are

Alternative	$\mu_{\tilde{I}}(x_i)$
1	.95
2	1
3	.77

The fuzzy set $\tilde{P}_2(p)$ indicating the degree to which alternative 2 is preferred to all others is shown in figure 14–11. p_2 is calculated as $p_2 = \bar{r}_2 - \frac{1}{2}(\bar{r}_1 + \bar{r}_3)$.

Many other fuzzy methods and models have been suggested to solve the MADM problem. They differ by their assumptions concerning the input data and by the measures used for aggregation and ranking. Also, they concentrate either on the first step (aggregation of ratings), or the second step (ranking), or both. Obviously all of them have advantages and disadvantages. They will, however, not be discussed here but will be in the second volume.

An interesting example of a more engineering-type application of multicriteria decision making using fuzzy sets is described by Muñoz-Rodriguez and Cattermole [1987].

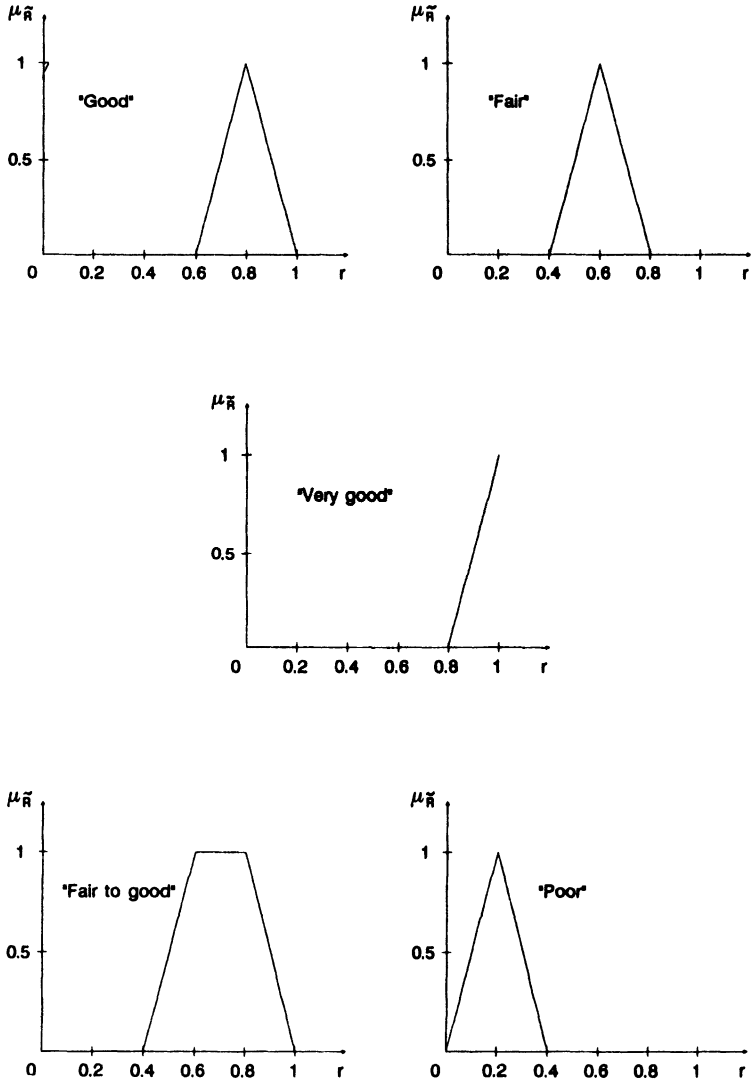


Figure 14–9. Fuzzy sets representing weights and ratings.

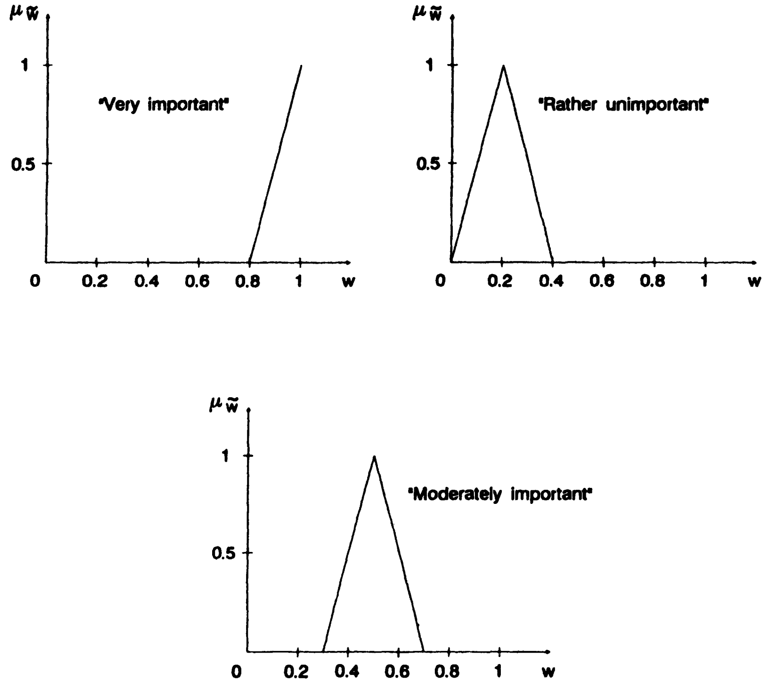


Figure 14–9. Continued.

Table 14–1. Ratings and weights of alternative goals.

Goal g_j	*Weight* $\tilde{w}_j$	*Rating $\tilde{r}_{ij}$ for alternative x_i* $i = 1$	2	3
1	very important	good	very good	fair
2	moderately important	poor	poor	poor
3	moderately important	poor	fair to good	fair
4	rather unimportant	good	not clear	fair

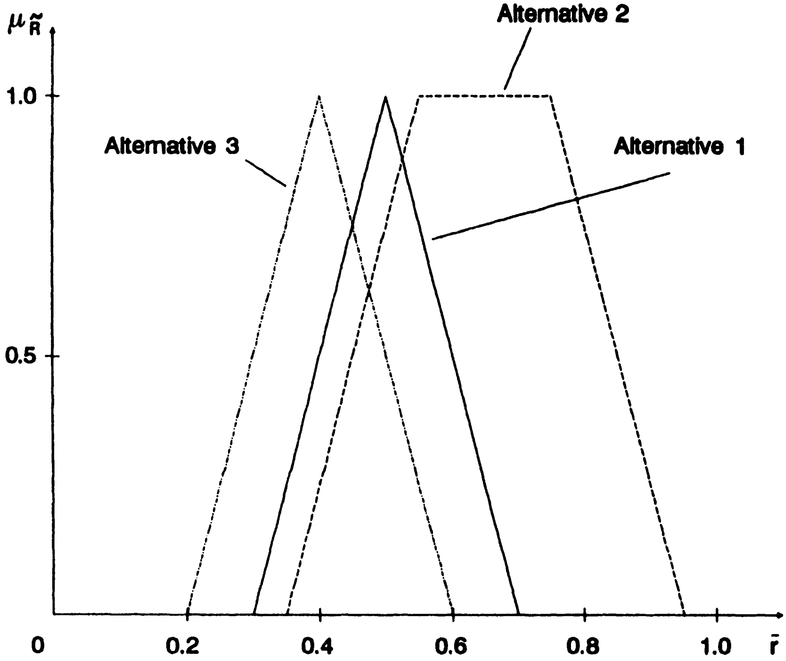


Figure 14–10. Final ratings of alternatives.

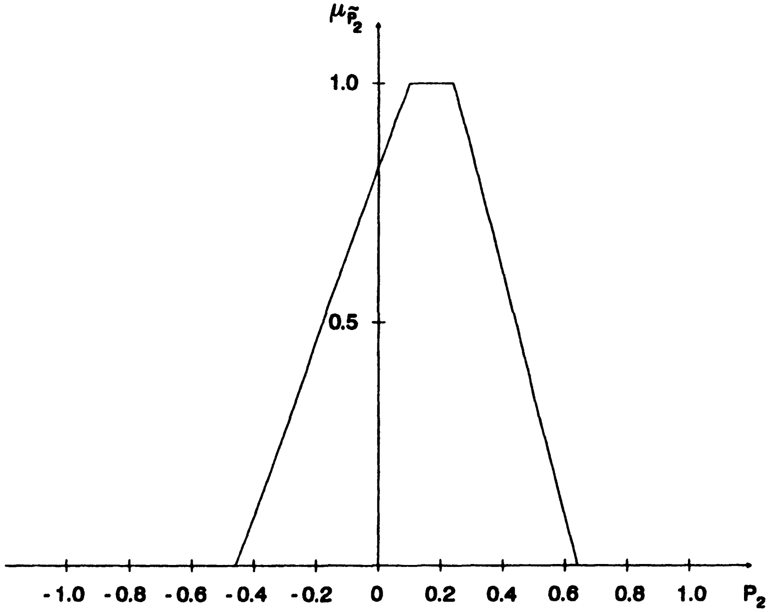


Figure 14–11. Preferability of alternative 2 over all others.

Exercises

1. Explain the (mathematical) difference between the symmetric and nonsymmetric model of a decision in a fuzzy environment.
2. Consider example 14–4. What grade would the student get if the "and" was interpreted as the "bold intersection" (definition 3–6), the "bounded difference" (definition 3–8), or the "bold union"?
3. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & z = 4x_1 + 5x_2 + 2x_3 \\ \text{such that} \quad & 3x_1 + 2x_2 + 2x_3 \leq 60 \\ & 3x_1 + x_2 + x_3 \leq 30 \\ & 2x_2 + x_3 \geq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Determine the optimal solution. Now assume that the decision maker has the following preferences:

a. He has a linear preference function for the objective function between the minimum and 1.5.
b. The tolerance intervals can be established as

$$p_1 = 10,\ p_2 = 12,\ p_3 = 3$$

Now use model (14.9) to determine the optimal solution and compare it with the crisp optimal solution.

4. Solve the example of exercise 3 by assuming the objective function to be crisp and by using equation (14.18).
5. Consider the problem:

$$\text{“maximize”} \quad Z(x) = \begin{Bmatrix} -x_1 - 3x_2 \\ 1.5x_1 + 2.5x_2 \end{Bmatrix}$$

$$\begin{aligned} \text{such that} \qquad -x_1 + 2x_2 &\le 18 \\ 4x_1 + 3x_2 &\le 40 \\ 3x_1 + x_2 &\le 25 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Determine an optimal compromise solution by using the model from example 14–10 (continuation).

6. What is the optimal alterative in the following situation (use Yager's method!)?

Alternatives: $X = \{x_1, x_2, x_3, x_4\}$
Goals: $\tilde{G}_1(x_i) = \{(x_1, .8), (x_2\ .6), (x_3, .4), (x_4, .2)\}$
$\tilde{G}_2(x_i) = \{(x_1, .4), (x_2\ .6), (x_3, .6), (x_4, .8)\}$
$\tilde{G}_3(x_i) = \{(x_1, .6), (x_2\ .8), (x_3, .8), (x_4, .6)\}$

The relative weights of the goals have been established as: $G_1 : G_2 : G_3 = 1 : 4 : 6$.

15 APPLICATIONS OF FUZZY SETS IN ENGINEERING AND MANAGEMENT

15.1 Introduction

The scope of applications of fuzzy sets—increasingly together with neural nets—is very large and still growing continuously. The closer the problem is to human evaluation, intuition, perception, and decision making, the less dichotomous is the problem structure and the more relevant and promising is the application of fuzzy technology.

In addition, one should realize, that we have moved from a situation of lack of computer readable data to a situation of an abundance of data, in which human beings are often unable to detect in the masses of available data the information that is relevant and valuable to them. Obviously there exists an increasing need for reduction of complexity by compactification of data. This is the reason for the increasing importance of (intelligent) data mining methods and tools. Web-technology is just opening completely new areas of application. If a model of a real problem does not consist of crisply defined mathematical statements and relations—if it is, for instance, a verbal model or a model containing fuzzy sets, fuzzy numbers, fuzzy statements, or fuzzy relations—then traditional mathematical methods cannot be applied directly. Either fuzzy algorithms—that is, algorithms that can deal with fuzzy entities or algorithms the procedure of which is "fuzzily"

H.-J. Zimmermann, *Fuzzy Set Theory - and Its Applications*

described—can be applied or one has to find crisp mathematical models that are in some specific sense equivalent to the original fuzzy model and to which available crisp algorithms can then be applied.

All cases in which fuzzy set theory is properly used as a modeling tool are characterized by four features:

1. Fuzzy phenomena, relations, or evaluations are modeled by a well-defined and founded theory. (There is nothing fuzzy about fuzzy theory!)
2. By doing so, a better approximation of real phenomena by formal models is achieved.
3. A better modeling of real phenomena normally requires more and more detailed information—more, in fact, than is needed for rather rough dichotomous modeling.
4. The amount of computer readable data is too large to be comprehended by a human observer.

When talking about "applications", different things can be meant:

1. One can "apply" one theory to another: for instance, one can apply fuzzy set theory to linear programming, which yields another theory, namely, fuzzy linear programming.
2. One can apply one theory to a model, which is an abstract picture of a possible real problem situation: the application of fuzzy set theory to inventory models, for instance, represents such an application.
3. One can apply a theory or a model to a real problem and solve it as well as possible.

We have considered applications of the first kind in chapters 9, 10, and partly in chapter 12, 13 and 14. This chapter is dedicated to applications of type 2 and 3, where often the existence of an application of type 2 triggers one of type 3.

The theory of fuzzy sets has already been applied to quite a number of operations research problems. As can be expected for a theory of this age, the majority of these "applications" are applications to "model problems" rather than to real-world problems. Exceptions are the areas of classification (structuring), control, logistics, and blending. For these areas there is already considerable software commercially available. The same is true for planning languages (decision support systems), for instance, in the area of financial planning. The reader should realize that the lack of real applications cannot necessarily be blamed on the theory. A real application of a certain theory normally requires that the practitioner who has the problem to be solved is also familiar with and understands, or at least accepts, the theoretical framework of the theory before it can really be applied. This obviously takes some more time.

Real applications, particularly the commercially successful ones, very often either are not published or are published after a long delay. This is partially due to competitive considerations and partly to the fact that practitioners normally do not consider publications as one of their prime concerns.

Nevertheless, for a textbook and for practitioners applications of type 2 and 3 are important since, as already mentioned, the knowledge of a type 2 application may trigger either other type 2 applications or type 3 applications.

The number of disciplines in which fuzzy sets are applied is increasing steadily. So far the main areas are (in alphabetical order and not in order of importance): actuarien science, business administration and management, chemistry, earth sciences, ecology and environmental science, economics, engineering (civil, industrial, mechanical, nuclear etc.), ergonomy, information technology, medicine, social sciences, telecommunication, traffic management.

It would obviously exceed the scope of this text book to cover the majority of these areas. Therefore, two areas were selected, which exhibit probably most applications: engineering and management. Fuzzy applications in these areas are increasingly known by the terms "business intelligence" and "engineering intelligence". Table 15–1 shows which applications are described in various chapters of this book.

15.2 Engineering Applications

Many, if not most engineering applications of fuzzy sets use the principle of fuzzy control that was studied in chapter 11. Hence, applications of this type were already described in the fuzzy control chapter. A second and large class of applications are located in (static) data analysis and hence, were studied in chapter 13. There are, however, numerous engineering applications which use other features or methods of fuzzy set theory. Examples of those can be found in [Kno and Cohen 1998], [Levner et al. 1998], [Jones and Hua 1998], [Gasos and Rosetti 1999], [Chen et al. 1998]. Some detailed descriptions and surveys can also be found in [Zimmermann 1999].

Most of these applications require too long a description to be included in this textbook. In order to describe the essentials of non-fuzzy control engineering applications we have chosen two examples: one showing a linguistic multicriteria analysis and one which uses dynamic fuzzy pattern recognition as described in chapter 13.

Table 15–1. Selected Applications in management and engineering.

Fuzzy Methodological Approach / *Functional Areas*	*Linear and Nonlinear Programming*	*Combinatorial Programming*	*Dynamic Programming Branch + Bound*	*Fuzzy Control + Appr. Reasoning*	*Graph. Theory*	*Clustering*	*General Fuzzy Math.*	*Other Heuristics*
Strategic Planning	×			×				
Media Selection	×							
Market Segmentation						×		
Creditworthiness				×			×	
Personnel Mgt.	×							
Location		×						
Transportation	×							
Production- and Process Control	×			×				×
Scheduling	×	×			×			×
Maintenance							×	
Inventory Control	×		×	×	×			
Quality Control						×		
Fault Detection							×	
Blending	×							
Assignment	×			×				
Engineering Design							×	

15.2.1 Linguistic Evaluation and Ranking of Machine Tools [Devedzić and Pap 1999]

The approach suggested by Devedzić and Pap is much broader applicable. It shall be illustrated, however, using a central part of these experimental study:

A metal cutting process generally is preceded by the following planning cycle:

(a) selection of processing method,
(b) operations selection and sequencing,
(c) machine tools selection,
(d) tooling selection,
(e) machining parameters selection and determination,
(f) tool path determination and calculation,
(g) NC programming, and
(h) cost and process economy calculations.

We shall concentrate on the machine tool selection. In particular, we shall focus on the machine tools rigidity modeling.

Rigidity has its clear mechanical definition and can be precisely determined for each machine tool element as well as for machine tools as a system in a whole. However, this approach is often applied only in the design stage, and during the exploitation period this characteristic is qualitatively evaluated as "high rigidity", "medium rigidity", "low rigidity", etc. These linguistic values are provided by skilled personnel based on experience, intuition and/or recently presented evidence. Metal cutting is a highly dynamical process which is influenced by numerous influences. The ultimate goal of a machining process is to produce a workpiece respecting the requested dimensional accuracy and surface quality.

One of the main characteristics of machine tools is their capability to reach workpiece requirements. On the other hand, evaluation of the machining process can be shown through realized productivity and economy. In reality, machine tools characteristics and output features could be indirectly perceived and represented through machine tool reigidity, which is often used as an integral qualitative feature of machine tool condition and capability. For linguistic modeling of machine tools rigidity Devedzić and Pap have used literature and empirical information and data, and results of an experiment conducted in laboratory.

Table 15–2 shows the measured data on which the experiment is based and table 15–3 shows the surface quality parameters that can be reached with four lathes, B, D, D and E.

Figure 15–1 depicts the terms of the linguistic variable "rigidity", where "norm(IT)" stands for a normalized surface quality.

Table 15–2. Experimental data.

Machine	Turret lathe
Cutting tool	PTGNR 2525 M16
Workpiece/material	Alloyed steel bar ($D = 52\,\text{mm}$, $L = 500\,\text{mm}$)/AISI 8620
Cutting speed υ (m/min)	80–350
Cutting feed s (mm/rev)	0.1–0.5
Cutting depth δ (mm)	1–3

Table 15–3. Surface quality parameters (output data).

Lathe	R_a (μm)	R_{max} (μm)	R_z (μm)	*Cut. depth*	$R_{a_{rel}}$ (%)	$R_{m_{rel}}$ (%)	$R_{z_{rel}}$ (%)
B	1–8	6–60	5–19	δ_1	83	68	35
C	1–8	7–39	4–20	δ_2	89	65	71
D	0.6–7	3–37	3–13	δ_3	79	74	52
E	1–10	9–39	5–25	δ_4	67	76	78

Note: $R_{(\bullet)rel} = R_{(\bullet)min}/R_{(\bullet)(B,C,D,E)}$; $R_m \equiv R_{max}$; $\delta_1 = 1\,\text{mm}$, $\delta_2 = 2\,\text{mm}$, $\delta_3 = 2.5\,\text{mm}$, $\delta_4 = 3\,\text{mm}$; R_a—arithmetical average deviation from center line; R_{max}—maximum height of surface roughness (microirregularities); R_z—mean height of surface roughness (microirregularities).

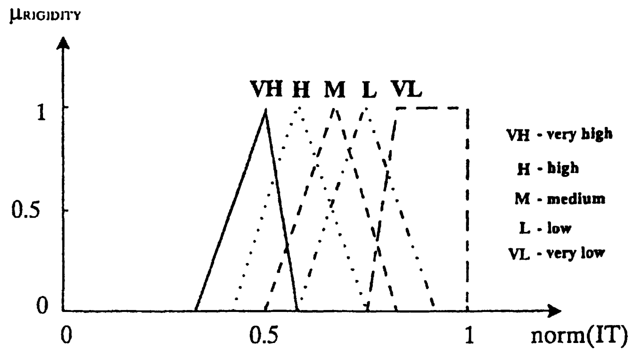


Figure 15–1. Linguistic values for variable “rigidity”.

The membership functions were checked empirically and proved to be acceptable. They are represented by the following triangular or trapezoidal membership functions:

$$\mu_{VH} = [.33, .5, .5, .58]$$
$$\mu_{H} = [.42, .58, .58, .75]$$
$$\mu_{M} = [.5, .67, .67, .83]$$
$$\mu_{L} = [.58, .75, .75, .92]$$
$$\mu_{VL} = [.75, .83, 1, 1]$$

Total machine tool rigidity depends on partial rigidity of all its elements. Among them the greatest influence performs three basic assemblies: main spindle, tailstock center, and toolholder.

It is clear that the significance of each of these elements is not the same. Therefore, it is quite useful to create a procedure for the generation of linguistic values mentioned above (figure 15–1), based on partial linguistic evaluation of rigidity value and significance of each element for given machining conditions.

Interviews with experts showed that in the case of partial evaluation of elements' rigidity values usually three linguistic values have been used (figure 15–2). Trapezoidal fuzzy numbers representing values of the linguistic variable "element rigidity" are defined as

$$\mu_{HS} = [.35, .35, .7, .8]$$
$$\mu_{MS} = [.7, .8, .8, .9]$$
$$\mu_{LS} = [.8, 1, 1, 1]$$

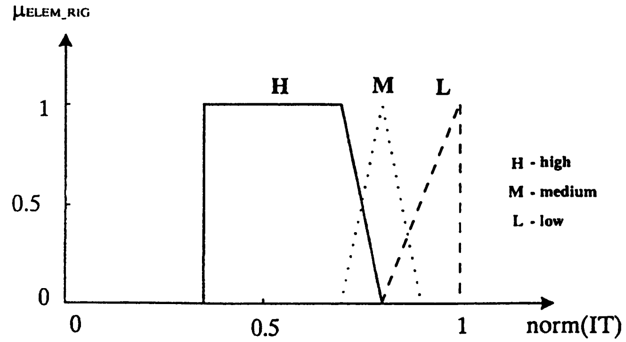


Figure 15–2. Linguistic values for variable "elements' rigidity".

Furthermore, using fuzzy sets representing values of variable elements' rigidity and empirical rules defining total machine tools rigidity, fuzzy sets determining significance of machine tools elements' rigidity have been defined (figure 15–3). Total number of empirical rules is very large and depend on cardinality of input term-sets.

For the determination of the values of the linguistic variable "significance", however, realistic boundary conditions allow the reduction of the rules to those shown in table 15–4:

The membership functions of the terms of the linguistic variable "significance", high, medium and low, were defined as:

Table 15–4. Boundary values of the linguistic variable "significance".

Element	*Value*	*Significance*	*Rigidity*
S	H	H	VH
TC	H	H	
TH	H	H	
S	H	M	VH
TC	H	M	
TH	H	M	
S	H	L	VH
TC	H	L	
TH	H	L	
S	M	H	M
TC	M	H	
TH	M	H	
S	M	M	M
TC	M	M	
TH	M	M	
S	M	L	M
TC	M	L	
TH	M	L	
S	L	H	L
TC	L	H	
TC	L	H	
S	L	M	L
TC	L	M	
TH	L	M	
S	L	L	VL
TC	L	L	
TH	L	L	

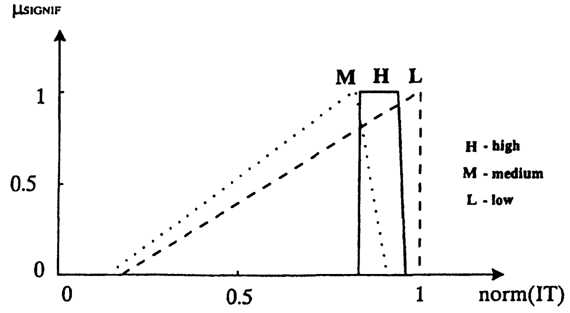


Figure 15–3. Linguistic values for variable "significance".

$$\mu_{HS} = [.9, .9, .965, .978]$$
$$\mu_{MS} = [.49, .894, .894, .949]$$
$$\mu_{LS} = [.512, 1, 1, 1]$$

They are shown in figure 15–3:

The determination of a linguistic value of rigidity is based on the above "dictionaries" of linguistic terms for "elements' rigidity" and "significance". The aggregating of element evaluations to an evaluation for each lathe is performed by using a weighted average of the element evaluations, i.e. a type of aggregation that was already mentioned in chapter 14 in multi attribute decision making. For each lathe j = {B, C, D, E} the evaluation is

$$\mu_j = \sum_i \mu_{S_i} \mu_{R_{ij}},$$

where index i denotes the three elements:

i = 1 = rigidity of main spindle
i = 2 = rigidity of tailstock center
i = 3 = rigidity of tool holder.

Figure 15–4 shows the results for the four lathes:

The ranking of the four fuzzy sets characterizing the lathes can now be performed by using any of the methods mentioned in chapter 14 or, for instance, by any of the methods compared by [Bortolan and Degain 1998]. The authors of this experiment use the center-of-gravity defuzzification and arrive at the order {B, D, E, C}.

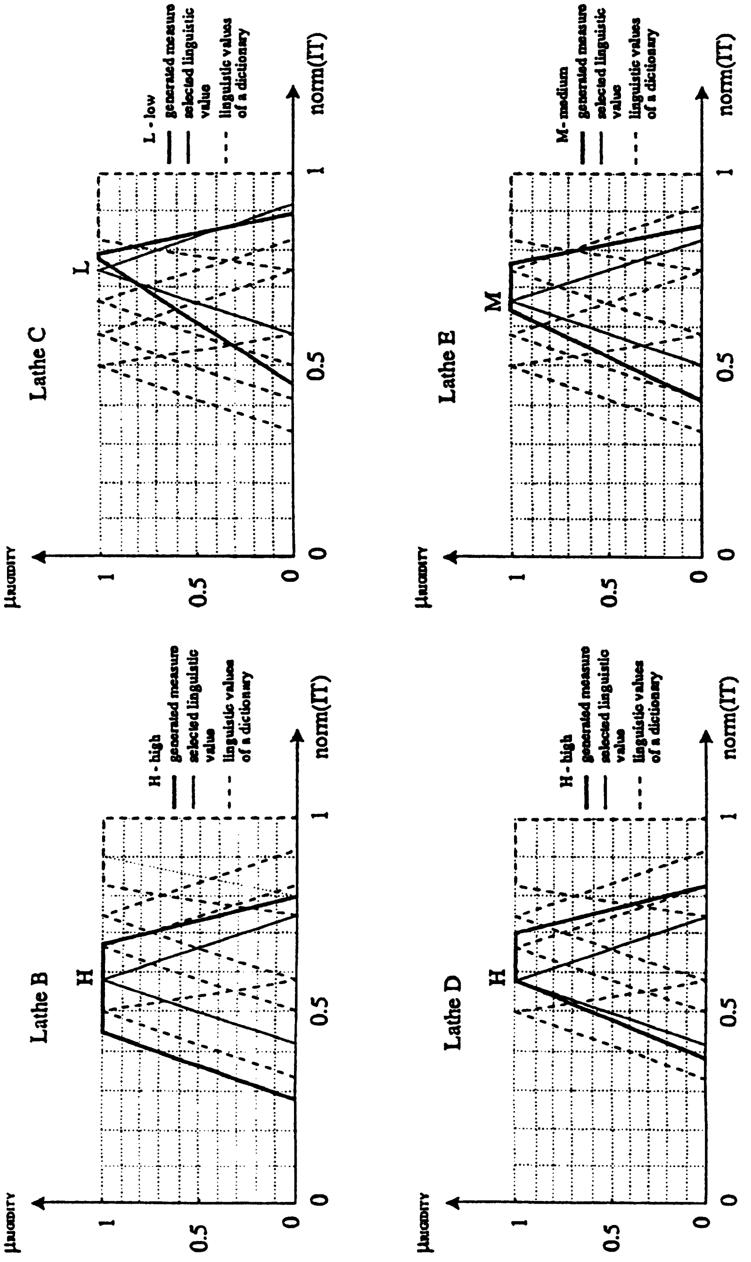


Figure 15–4. Linguistic evaluation values of lathes B, C, D, E.

15.2.2 Fault Detection in Gearboxes [Joentgen et al. 1999]

The problem is that of automatic fault detection in gearboxes (in this case of a helicopter gearbox). Used are the methods of fuzzy dynamic data analysis described in section 13.3.

The state-dependent maintenance of machines is a strategy to increase the availability of machines and to simultaneously improve the planning of down times. One prerequisite is the precise and reliable monitoring of machine's states. Since continuous machine monitoring by a trained expert is very time consuming and expensive, various systems for automatic diagnosis have been developed. They have been successfully applied in different areas, e.g., for diagnosis of electric motors [Fogliardi 1997], tape deck chassis [Fochem, Wischnewski, and Hofmeier 1997], saw blades [Brandt et al. 1996], roller bearings [Fochem, Joentgen, and Geropp 1997], household appliances [Weber, Wischnewski, and Fochem 1997], household appliances [Weber, Wischnewski, and Fochem 1997], etc. All these diagnostic systems use vibration analysis [Geropp 1995] to detect faults in machine components. Vibration analysis is based on the examiantion of solid-born signals measured at different places of a machine during operation. Changes of a machine's state lead to changes in the vibration signal. Using either expert knowledge or preliminary knowledge, it can be decided whether a change in the machine's state is due to a fault or to changes in some operating parameters.

Many of the above mentioned diagnostic systems use classification methods such as fuzzy c-means [Bezdek 1981] and fuzzy Kohonen networks [Tsao, Bezdek, and Pal 1994] to recognize different states of a machine. These methods require a selection of features which are relevant for the recognition of faults. The feature selection is usually based on an expert's knowledge and is crucial for the classification results.

In this application the functional fuzzy c-means algorithm (FFCM), described in 13.3, is used for automatic fault detection in gearboxes based on measured vibration signals. This algorithm is suited for the classification of dynamic objects, i.e., objects described by trajectories of their features. This paper shows how to apply the FFCM algorithm for early recognition of state's changes as well as for feature selection without any requirements for expert knowledge.

The subject of investigation in this paper is an intact gearbox. It is observed over a period of time during operation under a constant load. Vibration signals are measured at different positions on the gearbox each minute.

The task of the analysis is to monitor the state of the gearbox and to recognize significant changes in its state based on the vibration signal.

The experiment covered a period of time of approximately 114 hours (6,830 minutes). To reduce the resulting data set, only every 10^{th} measurement was taken into consideration (i.e., minutes 1, 11, 21 etc.). Carrying out the analysis for translated data (e.g., minutes 5, 15, 25 etc.) does not change the final results.

During preprocessing for each point of time the vibration signal measured was converted into a frequency spectrum containing 1024 values using Fourier transformation.

Thus, at each of 682 points of time a frequency spectrum consisting of 1024 values was given for the analysis. The brightness of the points in the figure represents the measured amplitude of a certain frequency at a corresponding point of time. To better illustrate the given data set, the upper bound of the scaling was set to 1.0, although some amplitude values exceed this upper bound.

In the course of the experiment a defect has occurred in the gearbox. This defect finds expression in the higher frequency amplitudes in the time interval between 500 and 640. Due to this defect, the gearbox was turned off at point 630 and repaired. According to experts, first symptoms of the defect can be recognized retroactively from point 320 on provided that the whole figure of the data set is at expert's disposal.

In this application, the objects to classify are states of the gearbox. Each state is described by one feature, i.e., a frequency spectrum. Each frequency spectrum at a given point of time is considered as a trajectory. Thus, there are in total 682 objects or trajectories, which can be clustered using the functional fuzzy c-means. The resulting class centers, which are frequency spectra, represent then typical states of the gearbox. Depending on the time interval chosen for the analysis, two procedures of classifier design can be distinguished:

- Clustering of all states of the gearbox from the beginning of the experiment till the current point (incremental classifier design);
- Clustering of the n latest states of the gearbox (rolling classifier design).

The following sections discuss these two types of the design procedure in more detail. First some general remarks on the chosen approach are given.

At point t c_t classes, which are typical states, are known. Now a classifier with $c_t + 1$ classes is designed. The shapes of the membership functions are then used to decide, whether a new class has occurred. Typically, if there is no new class, two or more of the membership functions are almost identical. In the other case a new class is discovered and has to be labeled.

Incremental Classifier Design. During incremental design of the classifier, the total information about a machine's states obtained from the beginning of the experiment until the current moment is used. In the course of the experiment available information is constantly supplemented with new data. For each point of time a classifier is designed based on information available so far.

Since the fuzzy c-means, as well as many other clustering methods, has difficulty recognizing classes with a very different number of objects correctly, new states of a machine are recognized only if they have appeared very often or differ very much from known states. Therefore, changes of states are discovered rather lately.

In this application 682 objects or trajectories consisting of 1024 values were clustered using the functional fuzzy c-means. At the beginning of the experiment, differences between frequency spectra can hardly be found. These spectra constitute the only class "State: intact". In the following two classes will be looked for.

The first change in the state of the gearbox appeared when the gearbox was started anew at point 230 after it was turned off for a while. The changes in the frequency spectra (measured with the distance measure for trajectories described in [Joentgen, Mikenina, Weber, and Zimmermann 1999]) are so large that these new appearing spectra are recognized as a separate class, although their number is very small. From this time on two typical states "State: intact" and "State: new start" are known. Therefore, in the following three classes will be looked for.

The third class "State: defective" is recognized approximately from point 440 on. Degrees of membership of objects to this class exceed all other degrees of membership after point 340 (figure 15–5). This means that at point 440 a fault in operation is recognized retroactively, starting at point 340.

Such late fault detection compared to the expert's statement is due to the above mentioned drawback of the fuzzy c-means and can be explained by the large number of objects representing class 1 "State: intact", which were observed in the time interval from 0 to 300. It should be noticed that measurements at points which are far in the past are not significant for the current fault detection. Considering them in the computations deteriorates the classification results.

Rolling Classifier Design. To avoid the problems related to different sizes of classes, the analysis of machine's states can be carried out periodically using time windows each covering 100 points. This means that only the data of the last 100 points of time are used for classifier design and classification. The size of the time windows was chosen arbitrarily. If the windows are too large, older states which persist over a long time period prevent early recognition of new states, if they are too small, older states may be forgotten and the system tends to recognize new states too often. For the sake of simplicity, only each 10^{th} time window is considered.

In the following the time window from point of time t_1 until point t_2 is denoted by $\langle t_1, t_2 \rangle$.

At the beginning of the experiment it is supposed that there exists only one class "State: intact". Therefore, the FFCM is used to look for two classes.

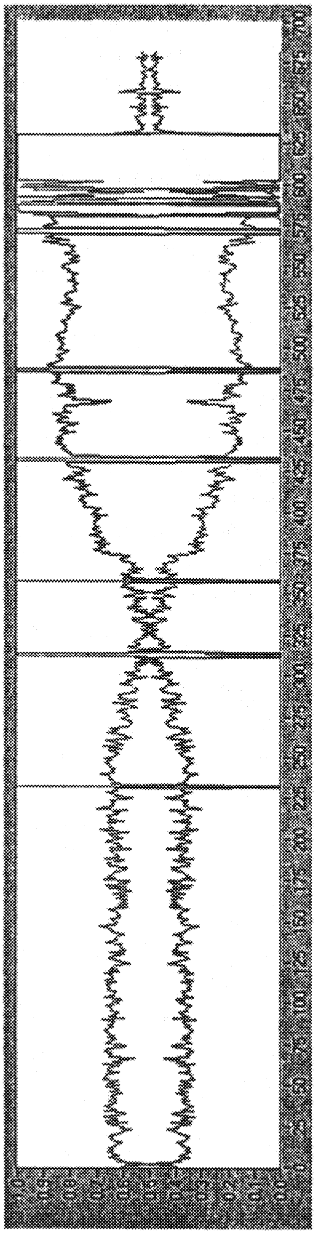


Figure 15–5. Membership functions resulting from incremental classifier design and classification of data obtained till point 440.

For time windows before window ⟨140, 240⟩ clustering results in two class centers, which can hardly be distinguished. Therefore, they can be considered as variations of class "State: intact".

Starting the gearbox anew at point 233, which appears at first in time window ⟨140, 240⟩, is recognized as a new class and labeled as "State: new start". Because now two typical classes of the gearbox are known, the algorithm will look for three classes in the following.

The new start of the gearbox at point 314, which is at first observed in time window ⟨220, 320⟩, leads to such large variations in the frequency spectra that the class "State: new start" is split into two classes. Therefore, from this point on four classes will be looked for.

In time window ⟨230, 330⟩ the fault is not recognized yet. Figure 15–6 shows the membership functions of the objects to the four calculated classes. Points at which the gearbox was started anew (points 233 and 314) can easily be recognized. At these points, degrees of membership of objects to class 3 are 1 whereas degrees of membership to the other two classes are 0. As stated above, restarting the gearbox at point 314 leads to the formation of two classes. Based on the membership functions shown in figure 15–6 it is not possible to distinguish clearly between classes 1 and 2. The centers of classes 1 and 2 are also almost identical. Therefore, it is assumed that classes 1 and 2 represent two parts of the class "State: intact".

Changes in the membership functions for classes 1 and 2 can first be recognized in time window ⟨240, 340⟩ (figure 15–7). After starting the gearbox anew at point 314 it is possible to distinguish between membership functions of classes 1 and 2. Considering the shapes of these functions in the whole time window, one can notice a decreasing and an increasing trend. The membership function

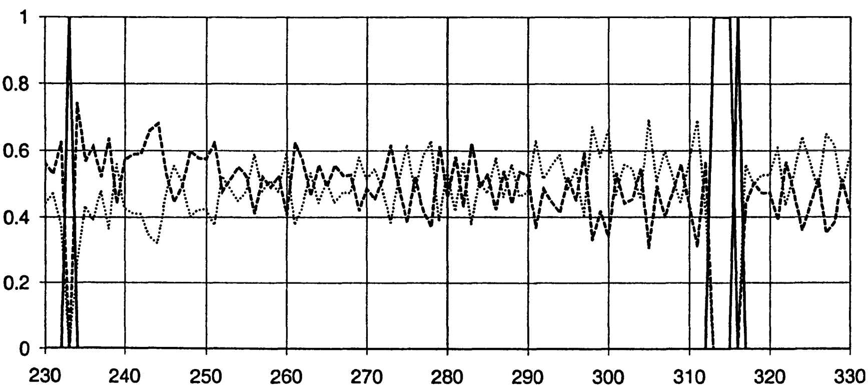


Figure 15–6. Membership functions for time window ⟨230, 330⟩.

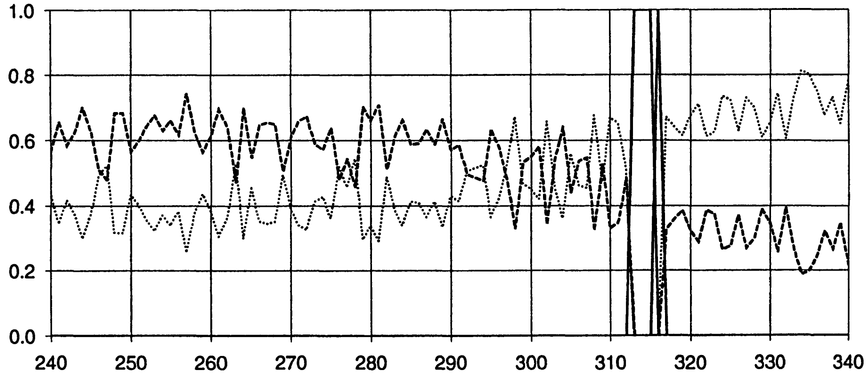


Figure 15–7. Membership functions for time window ⟨240, 340⟩.

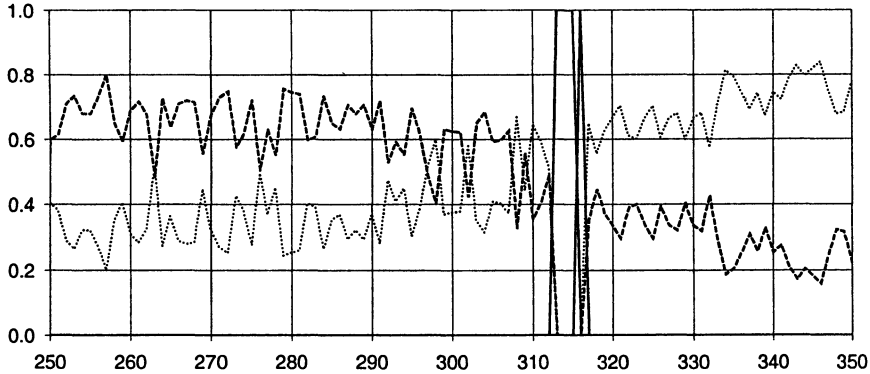


Figure 15–8. Membership functions for time window ⟨250, 350⟩.

with a decreasing trend corresponds to class 1 "State: intact" whereas the one with an increasing trend corresponds to class 2 "State: defective".

These trends are even stronger in the next time window ⟨250, 350⟩, shown in figure 15–8. Membership functions characterizing classes 1 and 2 can be distinguished even better than in figure 15–5 and figure 15–6.

The defect in the gearbox can be recognized from point 340 on. This defect is detected 20 points (i.e., 200 min.) later than it should be possible according to the expert's statement. It should be noticed that the diagnosis by an expert can also be carried out only retroactively. Thus, the functional fuzzy c-means allows an early on-line fault detection in the gearbox.

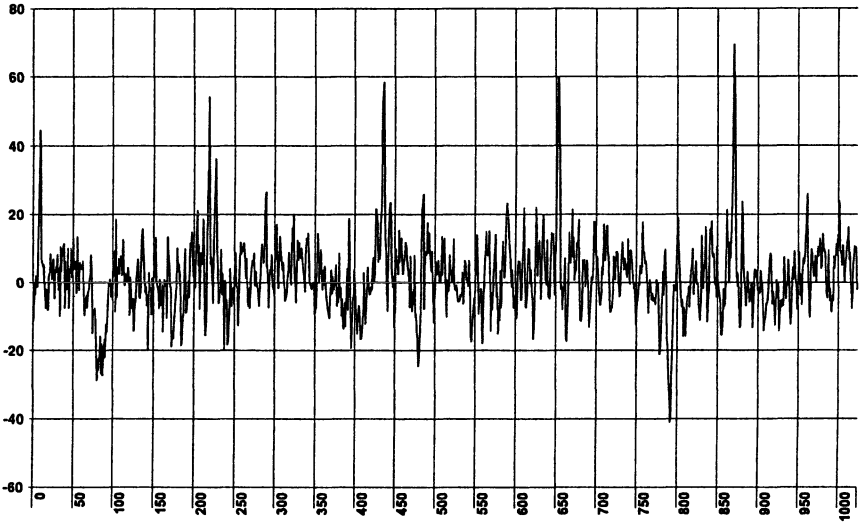


Figure 15–9. Proportional difference between class centers 1 and 2 (with respect to the center of class 2) in time window ⟨250, 350⟩.

Significant differences between the centers of classes 1 and 2 can be found at those spectral lines, at which bright vertical lines start or finish. These differences with respect to the center of class 2 are illustrated in figure 15–9.

Peaks in figure 15–9 correspond to characteristic frequencies, which are relevant for fault detection.

Refinement of the Analysis. The analysis described in the two previous sections was based on just 10% of the available data. In this section it will be investigated whether the use of all data can lead to an earlier detection of the change from "State: intact" to "State: defective". During on-line state-monitoring all these data would be available and could be taken into account.

As was shown above, the functional fuzzy c-means can recognize the defective state of the gearbox from point 314 on (i.e., from the 3140th minute on). For a more precise analysis, now the time interval from minute 2900 to minute 3300 is considered. To remain consistent with previous calculations, the analysis is carried out for time windows each covering 100 points.

The considered time interval contains 36 minutes (starting at point 3114), when the gearbox was turned off and started anew. This action represents an

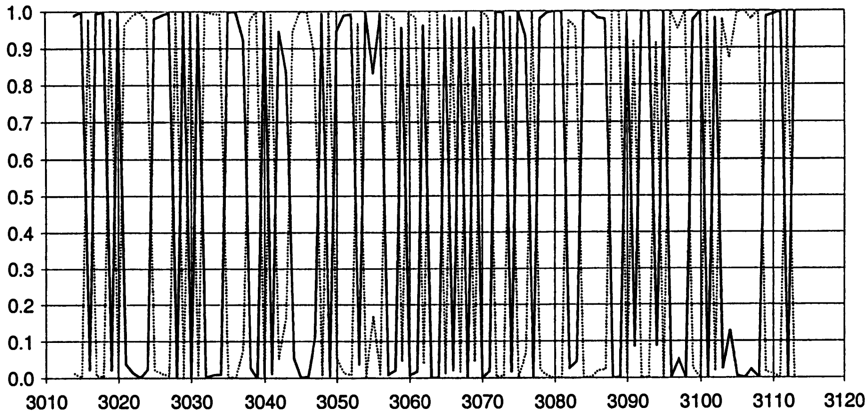


Figure 15–10. Membership functions for time window ⟨3014, 3114⟩.

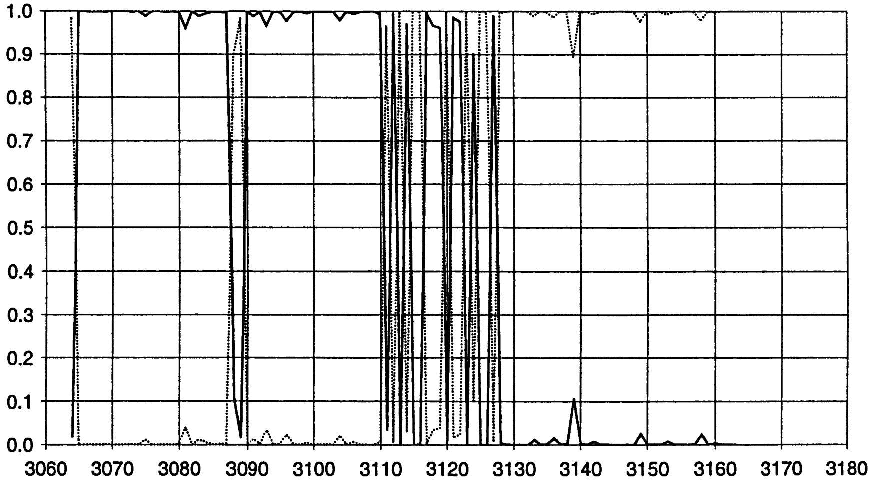


Figure 15–11. Membership functions for time window ⟨3064, 3200⟩.

external disturbance of the gearbox. The measured data are not relevant for a description of a gearbox's state during normal operation. Therefore they can be excluded from the analysis. The succeeding points of time are shifted backwards by 36 minutes. In the following, two classes will be looked for.

In time window ⟨3014, 3114⟩ (just before starting a gearbox anew) it is not yet possible to distinguish two classes. The corresponding membership functions are shown in figure 15–10.

Clear recognition of two classes is possible in time window ⟨3064, 3200⟩ (disregarding the 36 minutes which start at point 3114). The corresponding membership functions are illustrated in figure 15–11.

As described above, the fault could not be found using time window ⟨230, 330⟩, i.e., until minute 3300 the fault was not detected. Using all available data the fault is detected about two hours earlier (around minute 3200).

The application presented in this chapter shows that using the functional fuzzy c-means automatic fault detection in gearboxes can be successfully carried out based on vibration signals. Moreover, combining the rolling classifier design with this method allows an early fault detection. When typical states of the gearbox are recognized, they must be judged by an expert. Beyond that, the method does not require any expert knowledge.

15.3 Applications in Management

The borderline between engineering and management applications is fuzzy. Many of the functions (such as scheduling, maintenance, layout planning, simultaneous engineering, etc.), which are actually management functions, are performed by engineers. In universities they are also partly taught in management schools and partly in industrial engineering. The underlying mathematical structures differs very often. While engineering problems normally are characterized by nonlinear functuality and by fewer variables, management problems are generally modeled linearly and they are very large in terms of the number of variables and constraints. There are, of course, exceptions to this rule: there are, for instance, more problems with a combinatorial character in management than in engineering and they are certainly hard to solve. Recent problems that require data mining have increased in importance and they seem to be more relevant in the management area than in the engineering area. The reason may be, that very often engineering problems are more limited in scope while management problems generally have to take into consideration the entire enterprise for which masses of data are stored in data warehouses.

In the following we will present fuzzy set applications in main areas of management. The examples are selected in such a way, that the most important areas as well as the most important methodological approaches are covered. Typical managerial problems, such as the determination of creditworthiness, which are described in other chapters, are not included again.

15.3.1 A Discrete Location Model [Darzentas 1987]

For quite a number of years, there has been a widespread interest in location models. For specific types of these problems, excellent review papers exist. One of the most popular models is the "simple plant location model" (SPLP) for which, for instance, Krarup and Pruzan [1983] summarize the existing literature through the mid-1980s. In this paper, the authors also establish some relationships between SPLP, other location problems, set-covering problems, and integer programming. One of the problems, the discrete location problem (DLP), can be formulated as a set-covering problem and principally solved by pure zero-one programming algorithms. In this type of problem, a number of facilities are to be located at specific points within an area, according to precisely quantified criteria. This results in a districting, that is, a plan that shows where the facilities have to be located and what locations they serve. However, in many location problems, especially those associated with social policies, noncrisply defined criteria are used such as how "near" or "accessible" a facility is or how "important" certain issues are, etc. In these cases, a fuzzy sets approach is more appropriate.

In such a problem, the decision maker's main task is the identification and evaluation of criteria on the basis of which an optimum will be obtained. The choice of specific locations can only be based on questions like:

- How "far" should people travel to reach a service point?
- How "important" are "bad" and "good" roads and public transport?
- Is "homogeneity" of social class and income within a subset important?
- Is it "very unfair" to locate two major facilities in one point?

The fuzzy nature of the problem can be accepted and introduced at various stages in the analysis.

There are two major obstacles to finding "optimal" solutions to DLPs: It is necessary but difficult to define all possible covers, that is, subsets of locations, which have to enter even the crisp DLP-model. For readers who are not aquainted with this type of problem, the above-mentioned paper by Krarup and Pruzan or the work of Darzentas [1987, pp. 330–337] are recommended. The second problem is the "evaluation" of the covers in order to select the best one.

The aim of a location project is easy to state: find the "best" districting—which means that the objective itself is a fuzzy set. There may also be a number of restrictions, such as "the budget allows for approximately M facilities" or "it is preferable that village i serves village m," and vice-versa, or "it is very important that i and j belong to the same district," and so on. Hence constraints can be formulated as fuzzy sets.

In a crisp model, the determination of the optimal districting can be performed by using integer programming algorithms. If the problem is of reasonable size, heuristic versions have to be used.

In fuzzy DLPs, possibly even with multiple criteria, this approach is not possible. One could then use either fuzzy integer programming (see, for example, Fabian and Stoica [1984] or Zimmermann and Pollatschek [1984]), or one could try to reduce the number of possible districtings to a reasonable size by eliminating nonfeasible and dominated covers. The remaining covers could be evaluated with respect to relevant criteria (yielding a fuzzy set for each criterion) and then ordered in analogy to methods described in section 14.4.

Example 15–1

Consider the road network shown in figure 15–12, which is part of a real road network. The points 1, . . . , 4 represent villages whose populations are given in table 15–5a. The distances between the villages are given in table 15–5b. The problem is to optimally locate three facilities in order to serve (cover) each village with only one facility. This problem in its nonfuzzy form can be formulated as a set-partitioning problem. The fuzzy version of the problem can be formulated as a symmetric fuzzy-decision model (see definition 14–1).

Suppose the three covers shown in figure 15–13 are the only covers feasible due to crisp constraints, which are omitted here. In figure 15–13, the villages hosting a facility are hatched. For the determination of the "best" cover, the grades of membership of all three covers to every fuzzy criterion are rated. These ratings and the fuzzy criteria are given in table 15–6. In this example, the degrees of membership of the covers in the fuzzy set "decision" are obtained using the min-operator. These degrees imply an order on the set of covers. If a crisp decision

Table 15–5a. Populations.

Village	*Population*
1	1,100
2	650
3	1,350
4	730

Table 15–5b. Distances between villages.

	Miles			
	1	*2*	*3*	*4*
1	—	11	7	9
2	11	—	—	14
3	7	—	—	—
4	9	14	—	—

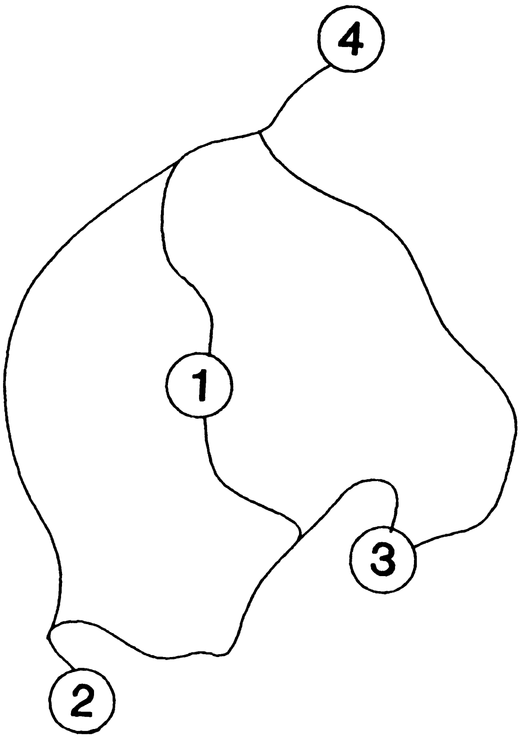


Figure 15–12. Road network.

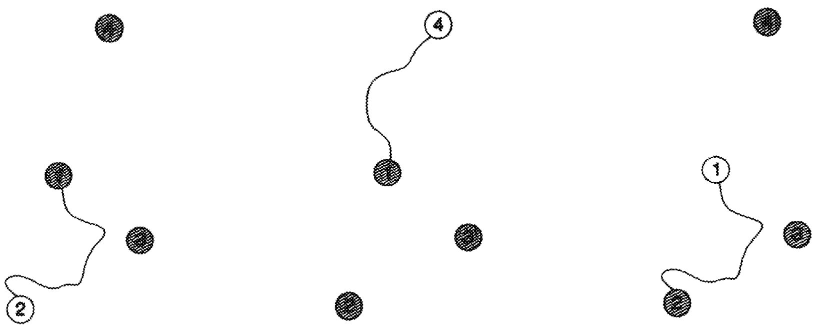


Figure 15–13. Feasible covers.

Table 15–6. Determination of the fuzzy set decision.

	Covers		
	c_1	c_2	c_2
It is a better policy to locate this type of facility in villages with high population:	.9	.8	.7
The facilities should not be located in polluted areas:	.6	.5	.2
The distance between a village without a facility and a facility should not exceed 8 miles considerably:	.6	.9	.6
Membership values of the decision:	.6	.5	.2

has to be made, the cover with the maximum degree of membership (c_1, $\mu_{\tilde{D}}(c_1) = .6$) is chosen.

15.3.2 Fuzzy Set Models in Logistics

OR has been applied extensively to the area of logistics in the past. In the following, two applications of fuzzy set theory are presented. At first, we show the "fuzzification" of a standard problem in OR: the transportation problem. Second—as an example of existing projects—we show a decision support system based on a fuzzy model.

15.3.2.1 Fuzzy Approach to the Transportation Problem [Chanas et al. 1984]. The analysis of "fuzzy counterparts" of linear programming problems of some special structure—for example, problems of flows in networks, transportation problems, and so on—appears to be an interesting task. The following model considers a transportation problem with fuzzy supply values of the suppliers and with fuzzy demand values of the receivers. For the solution of the problem, parametric programming is used.

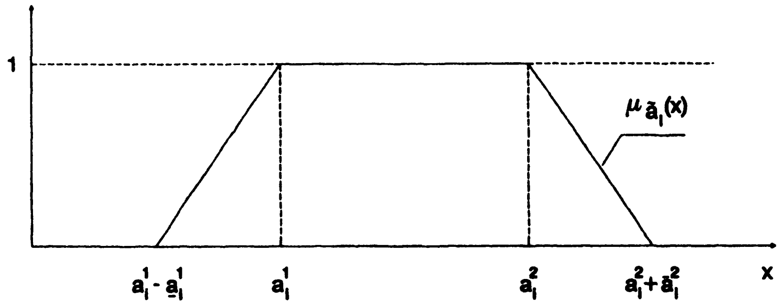


Figure 15–14. The trapezoidal form of a fuzzy number $\tilde{a}_i = (a_i^1, \underline{a}_i^1, a_i^2, \bar{a}_i^2)$.

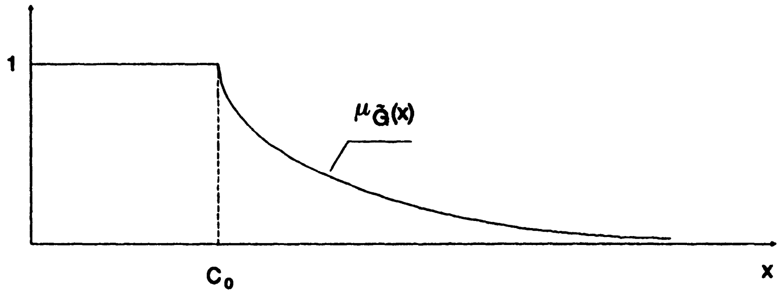


Figure 15–15. The membership function of the fuzzy goal $\tilde{G}$.

Model 15–1

$$\text{minimize} \quad c = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{such that} \quad \sum_{j=1}^{m} x_{ij} \cong \tilde{a}_i \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \cong \tilde{b}_j \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

$\tilde{a}_i$ and $\tilde{b}_j$ denote nonnegative fuzzy numbers of trapezoidal form. Note the slight difference between definition 5–3 and the definition shown in figure 15–14, which is only used for this section. The value of $\mu_{\tilde{a}}(\Sigma_j x_{ij})(\mu_{\tilde{b}}(\Sigma_i x_{ij}))$ is interpreted as a feasibility degree of the solution with respect to the ith (jth) constraint in model

15–1. With the objective function of model 15–1, a fuzzy number $\tilde{G}$ is associated, expressing the "admissible" total transportation costs. The membership function, $\mu_{\tilde{G}}$, of the $\tilde{G}$ is assumed to be of the form

$$\mu_{\tilde{G}}(x) = \begin{cases} 1 & \text{for} \quad x < C_0 \\ f(x) & \text{for} \quad x \geq C_0 \end{cases}$$

where $f(x)$ is a continuous function, decreasing to zero and achieving the value 1 for $x = C_0$ (see figure 15–15). In particular, $f(x)$ may be a linear function. $\mu_{\tilde{G}}(x)$ determines the degree of the decision maker's satisfaction with the achieved level of the total transportation costs.

Model 15–1 now can be reduced to the symmetrical decision model 15–2, assuming goal and constraints are aggregated via the min-operator.

Model 15–2

$$\begin{aligned} &\text{maximize} \quad \lambda \\ &\text{such that} \quad \mu_{\tilde{G}}(c(x)) \geq \lambda \\ &\qquad \mu_{\tilde{a}_i}\left(\sum_j x_{ij}\right) \geq \lambda \quad i = 1, 2, \ldots, m \\ &\qquad \mu_{\tilde{b}_j}\left(\sum_i x_{ij}\right) \geq \lambda \quad j = 1, 2, \ldots, n \\ &\qquad \lambda \geq 0 \quad x_{ij} \geq 0 \end{aligned}$$

Here, however, this problem shall be solved by parametric programming. For each level of a constraint's fulfillment λ, $\lambda \in [0, 1]$, one has to find the cheapest transportation plan. This plan satisfies the goal $\tilde{G}$ to the maximum degree for the respective λ. Hence in analogy to definition 14–5 and example 14–7, we shall determine

$$\max\{\mu_{\tilde{G}}(x) \wedge \mu_{\tilde{C}}(x)\}$$

where $\mu_{\tilde{C}}(x)$ will first be determined by an appropriate linear programming model. Here the min-operator is assumed to be acceptable. For the subsequent aggregation of $\mu_{\tilde{C}}(x)$ and $\mu_{\tilde{G}}(x)$, any nondecreasing operator and any decreasing function for $f(x)$ can be employed. Let us first turn to the determination of $\mu_{\tilde{C}}(x)$: The parameter of our parametric LP shall be denoted by r, $r \in [0, 1]$, and rather than determining λ-cuts we shall consider $(1 - r)$-cuts. Using the definition given in figure 15–14 for the fuzzy numbers specifying supplies and demands, the $(1 - r)$-cuts are intervals of the form:

$$\tilde{a}_i^{1-r} = \{x | \mu_{\tilde{a}_i}(x) \geq 1 - r\} = \left[a_i^1 - r\underline{a}_i^1, a_i^2 + r\bar{a}_i^2\right]$$
$$\tilde{b}_i^{1-r} = \{x | \mu_{\tilde{b}_i}(x) \geq 1 - r\} = \left[b_j^1 - r\underline{b}_j^1, a_j^2 + r\bar{b}_j^2\right]$$

Our problem can then be modeled as follows:

Model 15–3

$$\begin{aligned}
&\text{maximize} && \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
&\text{such that} && \sum_{j=1}^{n} x_{ij} \in \left[a_i^1 - r\underline{a}_i^1,\ a_i^2 + r\bar{a}_i^2\right] \quad i = 1, 2, \ldots, m \\
& && \sum_{i=1}^{m} x_{ij} \in \left[b_j^1 - r\underline{b}_j^1,\ b_j^2 + r\bar{b}_j^2\right] \quad j = 1, 2, \ldots, n \\
& && x_{ij} \geq 0 \quad r \in [1 - \bar{r}, 1]
\end{aligned}$$

where $\bar{r} = \sup_x \mu_{\tilde{a} \cap \tilde{b}}(x)$, that is, the maximum value of $\mu_{\tilde{C}}(x)$ that can be achieved for a given r. Solving this model either as a parametric LP or with special algorithms for parametric transportation models, we obtain $\mu_{\tilde{C}}(r)$ for $r \in [1 - \bar{r}, 1]$. This can now be combined with $\mu_{\tilde{G}}(r)$ to define the bership function of the fuzzy set "decision."

Example 15–2 [Chanas et al. 1984]

There are two suppliers with supply values:

$$\tilde{a}_1 = (10, 5, 10, 5) \text{ and } \tilde{a}_2 = (16, 5, 16, 5) \quad \text{(triangular fuzzy numbers)}$$

and three receivers with demand values:

$$\tilde{b}_1 = (10, 5, 10, 5), \quad \tilde{b}_2 = (9, 4, 9, 4); \quad \tilde{b}_3 = (1, 1, 1, 1)$$

(also triangular fuzzy numbers), respectively. The unit transport costs are

$$\begin{aligned}
c_{11} &= 10 \quad c_{12} = 20 \quad c_{13} = 30 \\
c_{21} &= 20 \quad c_{22} = 50 \quad c_{23} = 60
\end{aligned}$$

The membership function of the fuzzy goal is linear:

$$\mu_{\tilde{G}}(x) = \begin{cases} 0 & \text{for} \quad x \geq 800 \\ 1 & \text{for} \quad x \leq 300 \\ \dfrac{800 - x}{500} & \text{for} \quad x \in [300, 800] \end{cases}$$

Model 15–3 for this example becomes:

$$
\begin{aligned}
\text{minimize}\quad & c = 10x_{11} + 20x_{12} + 30x_{13} + 20x_{21} + 50x_{22} + 60x_{23} \\
\text{such that}\quad & x_{11} + x_{12} + x_{13} \geq 10 - 5r \\
& x_{11} + x_{12} + x_{13} \leq 10 + 5r \\
& x_{21} + x_{22} + x_{23} \geq 16 - 5r \\
& x_{21} + x_{22} + x_{23} \leq 16 + 5r \\
& x_{11} + x_{21} \geq 10 - 5r \\
& x_{11} + x_{21} \leq 10 + 5r \\
& x_{12} + x_{22} \geq 9 - 4r \\
& x_{12} + x_{22} \leq 9 + 4r \\
& x_{13} + x_{23} \geq 1 - r \\
& x_{13} + x_{23} \leq 1 + r \\
& x_{ij} \geq 0 \ \forall i, j
\end{aligned}
$$

Table 15–7 shows the parametric transportation problem table. Column FR denotes a "fictitious" receiver, row FD a "fictitious" supplier, and M a large real number. The rows and columns without an asterisk correspond to the suppliers having supply values settled at the minimum level. In this section the FD and FR are blocked by assigning a large transport cost M to their cells. The rows and columns with an asterisk correspond to the maximum surplus of the product that may be sent additionally (but not necessarily, and therefore the respective transport costs to the "fictious" receiver and suppliers are equal to zero) if the constraints are to be satisfied at least to the degree $1 - r$.

It should be observed that the joint supply value of all the suppliers is equal to $\tilde{a} = (26, 10, 26, 10)$ and the joint demand value of all the receivers is equal to $\tilde{b} = (20, 10, 20, 10)$. The maximum degree to which the constraints could be satisfied is equal to $\bar{r} = .7$. Therefore the relevant interval for analysis is $r \in [.3, 1]$.

Table 15–7. Table of the parametric transportation problem.

	Receivers							
Suppliers	*1*	*2*	*3*	*1**	*2**	*3**	*FR*	*Supply*
1	10	20	30	10	20	30	M	$10 - 5r$
2	20	50	60	20	50	60	M	$16 - 5r$
1*	10	20	30	10	20	30	0	$10r$
2*	20	50	60	20	50	60	0	$10r$
FD	M	M	M	0	0	0	0	$20r$
Demand	$10 - 5r$	$9 - 4r$	$1 - r$	$10r$	$8r$	$2r$	$6 + 20r$	

Table 15–8. Solution to transportation problem.

	$.3 \leq r \leq 1/3$	$1/3 \leq r \leq .6$	$.6 \leq r \leq 1$
x_{12}	$3 + 14r$	$9 - 4r$	$9 - 4r$
x_{13}	$7 - 19r$	$1 - r$	$1 - r$
x_{21}	$10 + 5r$	$10 + 5r$	$16 - 5r$
x_{22}	$6 - 10r$	$6 - 10r$	

The solution of this example is shown in table 15–8. The membership function $\mu_{\tilde{G}}(r)$ takes the form

$$\mu_{\tilde{G}}(r) = \begin{cases} .06 + 1.38r & \text{for} \quad r \in \left[.3, \frac{1}{3}\right] \\ .18 + 1.02r & \text{for} \quad r \in \left[\frac{1}{3}, .6\right] \\ .54 + 0.42r & \text{for} \quad r \in [.6, 1] \end{cases}$$

The maximizing solution is obtained for $r = .4059$ and $\mu_{\tilde{G}}(.4059) = .5941$. Figure 15–16 depicts this situation in analogy to figure 14–5.

15.3.2.2 Fuzzy Linear Programming in Logistics. Ernst [1982] suggests a fuzzy model for the determination of time schedules for containerships, which can be solved by branch and bound, and a model for the scheduling of containers on containerships, which results eventually in an LP. We shall only consider the last model (a real project).

The model contained in a realistic setting approximately 2,000 constraints and originally 21,000 variables, which could then be reduced to approximately 500 variables. Thus it could be handled adequately on a modern computer. It is obvious, however, that a description of this model in a textbook would not be possible. We shall, therefore, sketch the contents of the modeling verbally and then concentrate on the aspects that included fuzziness.

The system is the core of a decision support system for the purpose of scheduling properly the inventory, movement, and availability of containers, especially empty containers, in and between 15 harbors. The containers were shipped according to known time schedules on approximately 10 big containerships worldwide on 40 routes. The demand for container space in those harbors was to a high extent stochastic. Thus the demand for empty containers in different harbors could either be satisfied by large inventories of empty containers in all harbors, causing high inventory costs, or they could be shipped from their loca-

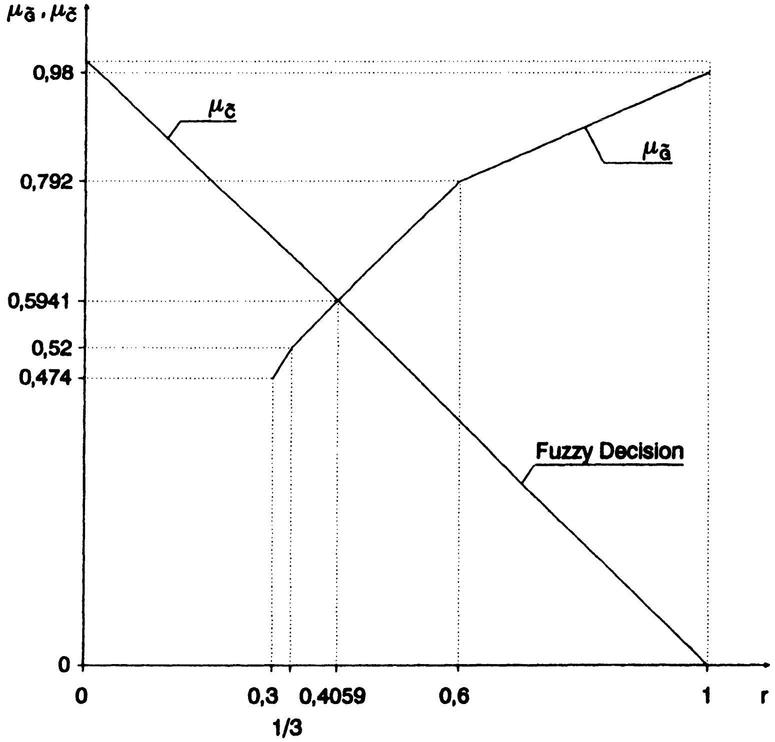


Figure 15–16. The solution of the numerical example.

tions to the locations where they were needed, causing high shipping costs and time delays.

Thus the system tries to control optimally primarily the movements and inventories of empty containers, given the demand in ports, the available number of containers, the capacities of the ships, and the predetermined time schedule of the ships.

This problem was formulated as a large LP model. The objective function maximized profit (from shipping full containers) minus cost of moving empty containers minus inventory costs of empty containers. When comparing data of past period with the model, it turned out that very often ships had transported more containers than their specific maximum capacity. This, after further investigations, led to a fuzzification of the ship's capacity constraints, which will be described in the next model.

Model 15–4 [Ernst 1982, p. 90]

Let

$z = c^T x$	the net profit to be maximized
$Bx \leq b$	the set of crisp constraints
$Ax \lesssim d$	the set of capacity constraints for which a crisp formulation turned out to be inappropriate

Then the problem to be solved is

$$\begin{aligned} \text{maximize} \quad & z = c^T x \\ \text{such that} \quad & Ax \lesssim d \\ & Bx \leq b \\ & x \geq 0 \end{aligned} \tag{15.1}$$

This corresponds to model (14.14). Rather than using model (14.19) to arrive at a crisp equivalent LP model, the following approach was used: Based on equation (14.10) and model (14.11), the following membership functions were defined for those constraints that were fuzzy:

$$\mu_i(t_i) = \frac{t_i}{p_i - d_i} \quad 0 \leq t_i \leq p_i - d_i \; i \in I,$$

I = Index set of fuzzy constraints.

As the equivalent crisp model to (14.1), the following LP was used:

$$\begin{aligned} \text{maximize} \quad & z' = c^T x - \sum_{i \in I} s_i (p_i - b_i) \mu_i(t_i) \\ \text{such that} \quad & Ax \leq d + t \\ & Bx \leq b \\ & t \leq p - b \\ & x, t \geq 0 \end{aligned} \tag{15.2}$$

where the s_i are problem-dependent scaling factors with penalty character.

Formulation (15.2) only makes sense if problem-dependent penalty terms s_i, which also have the required scaling property, can be found and justified.

In this case the following definitions performed successfully: First the crisp constraints $Bx \leq b$ were replaced by $Bx \leq .9b$, providing a 10% leeway of capacity, which was desirable for reasons of safety. Then "tolerance" variables t were introduced:

$$\begin{aligned} Bx - t &\leq .9b \\ t &\leq .1b \end{aligned}$$

The objection function became

$$\text{maximize } z = c^t x - s^t t$$

where s was defined to be

$$s = \frac{\text{average profit of shipping a full container}}{\begin{array}{l}\text{average number of time periods that elapsed}\\ \text{between departure and arrival of a container}\end{array}}$$

Because of this definition, more than 90% of the capacity of the ships was used only if and when very profitable full containers were available for shipping at the ports, a policy that seemed to be very desirable to the decision makers.

Before turning to another application area, it should be mentioned that other applications of fuzzy set theory can be found in the literature [Oh Eigeartaigh 1982] and that the development of model (14–9) was initially triggered by a real problem in logistics described by Zimmermann [1976].

15.3.3 Fuzzy Sets in Scheduling

Scheduling is a very common activity in management. It concerns very different areas, i.e. production, maintenance, transportation, activities, etc. The environments in these different areas differ from each other and so do the specific constraints that have to be taken into consideration. Some of these scheduling tasks have a more engineering character (such as in telecommunication, in repair, in computer networks, etc.). They are not considered here. We rather focus on those areas that are generally in the general management domain. Sometimes planning and scheduling (or control) are closely related to each other and can hardly be separated. Also, some neighboring areas, such as production- and inventory control are very much interrelated. We shall present six cases which cover different areas and also different approaches for planning and scheduling.

15.3.3.1 Job-Shop Scheduling with Expert Systems [Bensana et al. 1988]. In the following, we will present a job shop scheduling approach where concepts from the field of artificial intelligence and concepts of fuzzy set theory enrich traditional OR.

Different kinds of knowledge cooperate in the determination of feasible schedules. One kind of knowledge is represented by rules. Relevances of rules with respect to facts and goals are expressed by concepts of fuzzy set theory. First, we will sketch the system. Second, we will focus on the application of fuzzy set theory within the system.

The scheduling problem in a workshop can be stated as follows: Given a set of machines and technological constraints, and given production requirements expressed in terms of quantities, product quality, and time constraints expressed by means of earliest starting times and due dates for jobs, find a feasible sequence of processing operations.

A set of K jobs must be performed by a set of M machines. Each job k is characterized by a set of operations O_k assigned to machines on which they have to be performed. A schedule is described by means of a precedence graph, expressed by a set of pairs (O_i, O_j) denoting that O_i must precede O_j.

The system, implemented in LISP and named OPAL, consists of two planning modules—the "constraint-based" analysis module and the "decision-support" module—whose interaction is guided by a "supervisor" module. The supervisor module plays the role of the inference engine and guides the search process. The structure of the system is shown in figure 15–17.

The *constraint-based analysis* (CBA) module deals with a partial order of operations derived from the processing sequence of parts and the schedule in progress on one side and the time constraints for job processing on the other. By subsequent systematic comparisons of the existing precedence constraints, new precedence constraints are generated. This procedure stops in one of the following states:

success: A feasible and complete schedule is derived.

failure: Due to conflicting precedence constraints, a feasible schedule does not exist.

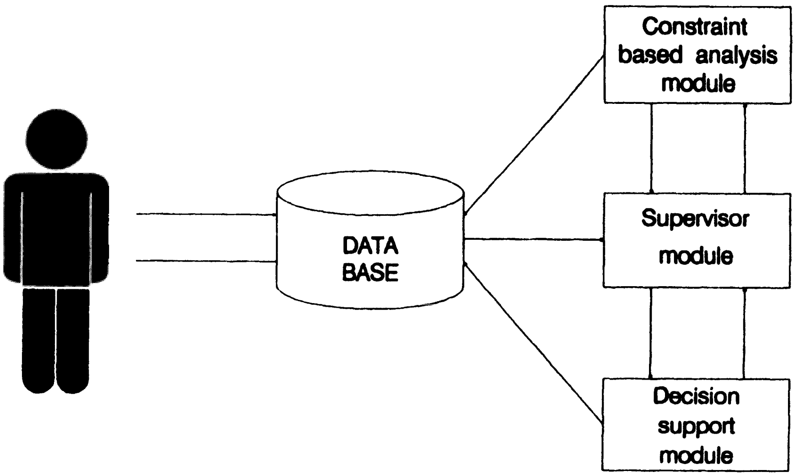


Figure 15–17. Structure of OPAL.

wait: The schedule in progress is incomplete (i.e., the set of precedence constraints does not form a complete order), and no more precedence constraints can be generated.

If the CBA module reaches a "wait" state, the decision pertaining to operation ranking is no longer dictated by feasibility considerations with respect to due dates. Such a decision can be made according to other kinds of criteria of a technological nature (e.g., it is better not to cut a workpiece made of metal M before a workpiece made of metal M'), or related to productivity (facilitate material flow, avoid filling up machine input buffers, avoid long set-up times . . .).

According to these criteria, a *decision-support module* (DS) generates new precedence constraints. First, it selects a subset C of the set of all unordered pairs of operations. Second, it choses one element of C and forms a new precedence constraint.

The selection can be based on criteria like specific machines, specific operations, temporal location, influence on the quality of the schedule, or influence on the resolution speed. The grades of membership of the unfixed pairs of operations in the sets defined by those criteria may be expressed fuzzily. If more than one criterion is used for selection, the corresponding fuzzy sets are intersected by the minimum-operator.

In the second step, one element of this fuzzy set is chosen to be fixed, that is, to be the new precedence constraint. This step is carried out by using a collection of pieces of advice expressed as "if . . . then" rules. Rules differ by their origin and by their range of application (general or application-dedicated). Moreover, their efficiency is more or less well known and depends upon the prescribed goal, or the state of completion of the schedule. These rules can express antagonistic points of view. Lastly, they are usually pervaded by imprecision and fuzziness, because their relevance in a given situation cannot be determined in an all-or-nothing manner.

To take these features into account, each rule r is assigned a grade of relevance $\pi_r(k)$ with respect to goal k. $\pi_r(k)$ can be viewed as the grade of membership of rule r to the fuzzy set of relevant rules for goal k. The aim of these coefficients is basically to create an order on the set of rules. For every pair of operations, the "if" part of a rule is evaluated as to the extent to which O_i should precede O_j according to the attribute of the rule. Let v be the index qualifying this attribute, and let v_{ij} be the value of this index when O_i precedes O_j. The ratio $x_{ij} = \dfrac{v_{ij}}{v_{ij} + v_{ji}}$ is then evaluated. To avoid thresholding effects, three fuzzy sets H = high ratio, M = medium ratio, and S = small ratio are defined (see figure 15–18). Hence the relation appearing in the rule is a fuzzy relation.

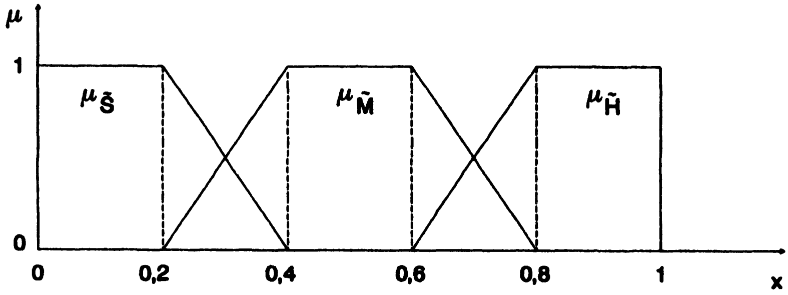


Figure 15–18. Fuzzy sets for the ratio in the "if" part of the rules.

The "then" part of all rules is of the same format. It provides advice about whether O_i should precede O_j $(i < j)$ or if the rule does not know $(i \sim j)$. This advice is expressed by three numbers:

$$\mu_r(i < j) = \min(\mu_{\tilde{S}}(x_{ij}), \pi_r(k))$$
$$\mu_r(j \sim i) = \min(\mu_{\tilde{M}}(x_{ij}), \pi_r(k))$$
$$\mu_r(j < i) = \min(\mu_{\tilde{H}}(x_{ij}), \pi_r(k))$$

The rules relevant for goal k are all triggered and applied to all facts in the set C. The proportions of relevant triggered rules preferring $i < j$, $j < i$, $i \sim j$ are obtained as relative cardinalities (see definition 2–50):

$$\rho(i < j) = \sum \mu_r(i < j) / \sum \pi_r(k)$$
$$\rho(j < i) = \sum \mu_r(j < i) / \sum \pi_r(k)$$
$$\rho(j \sim i) = \sum \mu_r(i \sim j) / \sum \pi_r(k)$$

When $\rho(i \sim j)$ is close to 1, it is not possible to decide which of the two operations should precede the other because the rules are indifferent. In contrast, when $\rho(i \sim j)$ is close to 0, but $\rho(i < j)$ is close to $\rho(j < i)$, the set of rules is strongly conflicting. The preference index for decision $i < j$ is defined as min $\{\rho(i < j), 1 - \rho(i \sim j)\}$; in terms of fuzzy logic, it expressed to what extent most of the triggered rules prescribe $i < j$, and most are not indifferent about O_i preceding O_j.

The schedule is gradually built up by adding precedence constraints between operations. The search graph is developed as follows: each time the CBA module stops, a new node is generated and the current schedule is stored. The DS module then generates a new precedence constraint to the schedule graph, and the CBA module checks for consequent precedence constraints. Backtracking occurs if the explored path leads to a failure state. When no feasible schedule at all exists, the data must be modified in order to recover feasibility.

15.3.3.2 A Method to Control Flexible Manufacturing Systems [Hintz and Zimmermann 1989]. The following application shows the usage of multiple concepts of fuzzy set theory within a hybrid system for production planning and control (PPC) in flexible manufacturing systems (FMSs). FMSs are integrated manufacturing systems consisting of highly automated work stations linked by a computerized material-handling system making it possible for jobs to follow diverse routes through the system (see figure 15–19). They facilitate small batch sizes, high quality standards, and efficiency of the production process at the same time.

Decentralized PPC systems for each FMS are provided with schedules of complete orders by an aggregate planning system. They are responsible for meeting the due dates, minimizing flow times, and maximizing machine utilizations. Generally, these objectives are conflicting. The planning process is carried out by subsequently solving the subproblems:

1. Master scheduling
2. Tool loading
3. Releasing scheduling
4. Machine scheduling

Subproblem 1 is solved by using fuzzy linear programming (FLP), subproblem 2 is solved by a heuristic algorithm, and subproblems 3 and 4 are solved using

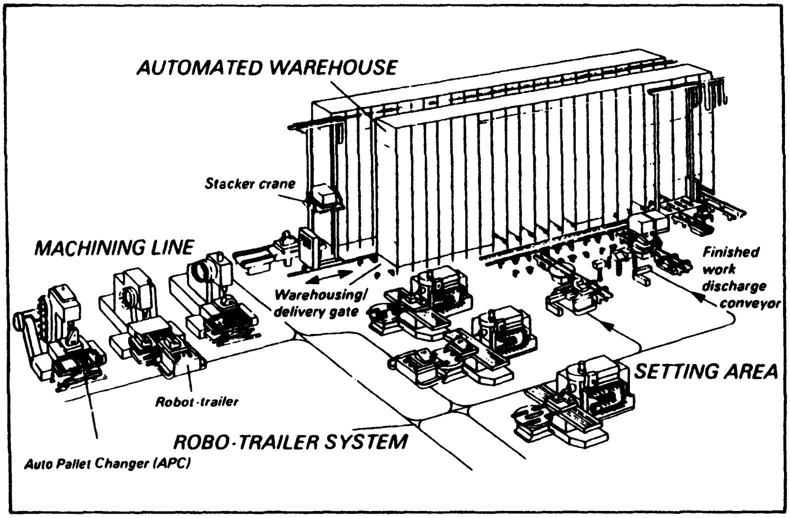


Figure 15–19. Example of an FMS [Hartley 1984, p. 194].

approximate reasoning (AR). We will just sketch the master scheduling, omit the tool loading, and focus on the release and machine scheduling.

Master Scheduling. The objective of the master schedule is to determine a short-term production program with a well-balanced machine utilization that optimally meets all due dates. Its determination is a quite well-structured problem, although some important input data are rather uncertain. Since nearly the complete manufacturing of a part can be performed within an FMS, a simultaneous approach using FLP (as defined in section 14.2.1) has been employed for the master scheduling. Restrictions to be considered in the master schedule are as follows:

1. Parts can only be processed when they are released from earlier production stages.
2. They have to meet given due dates in order to match the following operations and assembling.
3. The capacity of the FMS must not be exceeded. Because the machines may partially be substituted by each other, they have to be classified into appropriate groups.
4. There is only a limited number of (expensive) fixtures and pallets available.

In restrictions 1 and 2, release and due dates are often rough estimates that include safety buffers and unnecessary work-in-process inventories. In practice it is often possible to supply some parts earlier than initially planned (i.e., by overtime) or to violate the due dates only for a portion of an order (for instance, by lot-size splitting) without seriously disturbing processing or assembling. On the other hand, if release dates or due dates are chosen too stringently, there may be no feasible solution at all.

For these reasons, restrictions 1 and 2 are modeled as fuzzy constraints while restrictions 3 and 4 are modeled as crisp constraints. The solution of the FLP yields a solution

- that is feasible according to restrictions 1–4, if possible, or
- that minimizes the deviations from given due dates and distributes them uniformly among the different orders. The value of the maximized variable then denotes the degree of membership of the optimal solution in the set of feasible and optimal solutions.

Release and Machine Scheduling. Decisions concerning the parts schedule for both releasing and machining are arrived at by AR. This is considered to be an appropriate way to model a very complex situation with many interdependencies. The decision criteria are formulated in terms of production rules, which have been shown to lead to quite stable decisions. It will be shown later that this approach also leads at least to a very good compromise of the tree mentioned conflicting goals of scheduling. In addition, this method is very suitable for interactive decision making, where the decision maker can employ familiar linguistic descriptions of the situations.

The basic *release scheduling* procedure can be regarded as dispatching parts for a single capacity unit (the FMS) with several work stations: As long as unused working places and appropriate pallets with fixtures are available, new parts can be released into the FMS. Once the upper limit of parts has been reached, the remaining parts have to wait in a queue until one of the parts leaves the FMS. Then the decision of which part should be released next will be made using an AR procedure.

The *machine scheduling* procedure is very similar to dispatching when using priority rules. This means that no machine is allowed to wait if there is a part that can be processed on that machine. If there are several parts at a time waiting for a machine, then another AR procedure is used to choose a part from the waiting line.

For both AR procedures, a *hierarchy of decision criteria* is defined (see figure 15–20). This hierarchy corresponds to stepwise operationalizing the decision criteria until they can easily be used by the decision maker. On the other hand, such a hierarchy can be considered as the combination of elementary local-priority rules in a more comprehensive global-priority or decision rule. The single elements or concepts of the hierarchy may in general consist of arithmetic or linguistic terms. Both the hierarchy and the ways to make the concepts operational are heuristic in nature. Hence no optimal solution can be guaranteed.

Let us further concentrate on the criteria hierarchy depicted in figure 15–20a for the release scheduling. The decision of which part to release next mainly depends on date criteria of the parts under consideration or the impact of parts on machine utilization, or it may depend on some kind of external priority. For

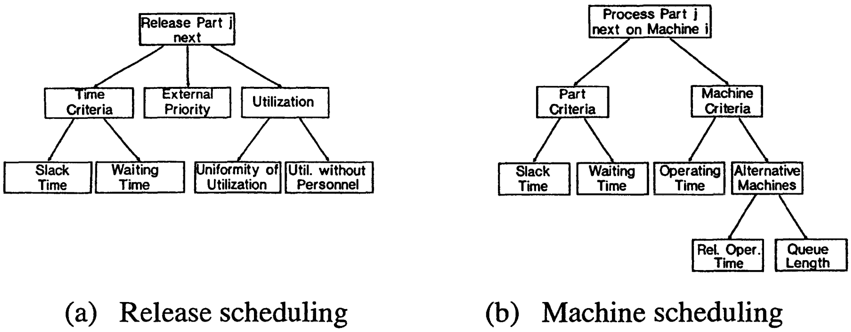


Figure 15–20. Criteria hierarchies.

the date criteria, we furthermore distinguish between the slack time of a part and the time the part has already waited for processing.

The impact on the effect on machine utilization can be twofold. First, we have to take care that the machines are used as uniformly as possible, thus trying to avoid bottlenecks. For this purpose, we define a criterion "uniformity of utilization." On the other hand, we want to ensure a good utilization in the shift with reduced personnel, during which no parts can be fixed on pallets. On the contrary, parts can only be processed as long as they do not need any manual operation, be it for changing a pallet or in any case of failure. We shall take this into consideration by using the concept "processing time until the next fixturing." The external priority can be given by the plant manager or some other person responsible.

To illustrate the AR process, we will look at the definitions of the concepts of the hierarchy and the aggregation of concepts by the rule set. We will focus on the derivation of the date criterion of the slack time and the waiting time criteria. Slack time and waiting time are considered linguistic variables as defined in section 9.1:

Linguistic variable	*Term set*
slack time	critically_short, short
waiting time	short, medium, long
date criterion	urgent, not_urgent

The base variable is defined for all possible values for the indicator, that is, in general, all real numbers within a reasonable interval. The meaning of the terms can be defined by giving the degree of membership as a function of the above-defined indicator as base variable. As membership functions, piecewise linear functions are used. The parameters were obtained by extensive simulation studies for a specific structure of orders to be processed in a specific FMS.

An essential task before aggregating these two criteria with the date criteria is the assignment of degrees of sensibleness to each element (rule) of the Cartesian product defined by the assumptions and the conclusion: {long, medium, short} $\otimes$ {critically_short, short} $\otimes$ {urgent, not_urgent}. This can be done by an expert (scheduler) and results in the "degrees of sensibleness" shown in parentheses for each rule above.

An example rule set might be (degree of sensibleness given in parentheses):

1. **IF** waiting time is long **AND** slack time is critically_short **THEN** date criterion is urgent (1.0)

2. **IF** waiting time is medium **AND** slack time is critically_short **THEN** date criterion is urgent (0.8)
3. **IF** waiting time is short **AND** slack time is critically_short **THEN** date criterion is urgent (0.6)
4. **IF** waiting time is long **AND** slack time is short **THEN** date criterion is urgent (0.5)
5. **IF** waiting time is medium **AND** slack time is short **THEN** date criterion is urgent (0.2)
6. **IF** waiting time is medium **AND** slack time is short **THEN** date criterion is not_urgent (0.7)

Each of these rules can now be interpreted as one possible aggregation of the two criteria "slack time" and "waiting time" with the "date criteria" (see figure 15–20a). Only rules with a nonzero degree of sensibleness are considered. The AR procedure applied is depicted in figure 15–21. That is, first the conditional

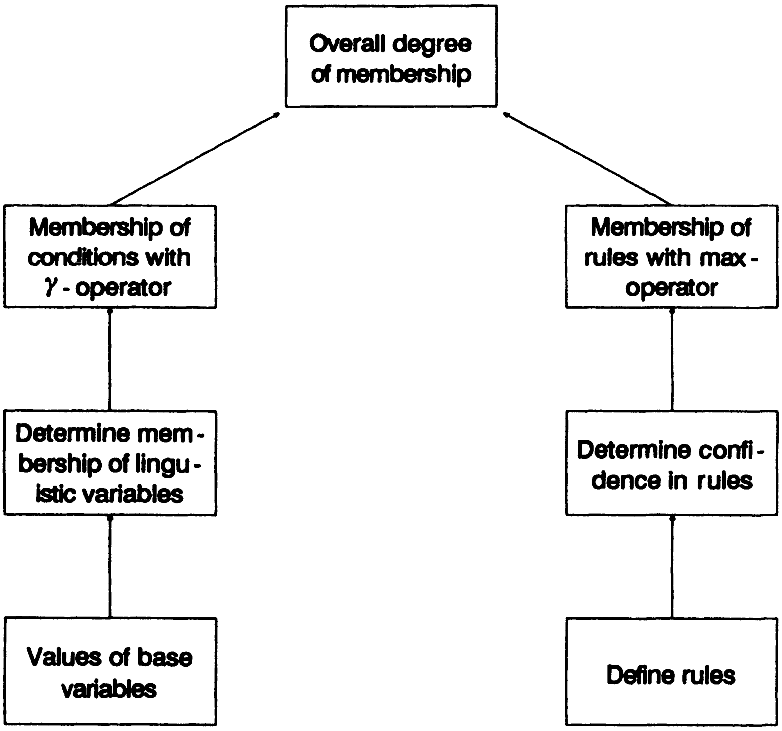


Figure 15–21. Principle of approximate reasoning.

parts of the rules connected by "AND" or "OR" are aggregated by using the γ operator. The "THEN" of the rule is then interpreted as "the conditions hold *and* the rule is valid," where this "AND" is also modeled by the γ operator. In this case, however, γ is taken to be zero, since no compensation is assumed between the truth of the rule and the validity of its conditions. If more than one rule leads to a certain condition, the maximum of the respective degrees of membership determines the final result.

Example 15–3

We want to compute the values (degrees of membership) of the terms of the "date criteria" in figure 15–20a. Consider three parts, whose slack time and waiting time are linguistic variables as described above. The grades of membership in terms of the linguistic variables are given in table 15–9.

In the first step, the conditional parts of the rules are aggregated by using the γ operator. In this example, $\gamma = .5$ is used. The results are depicted in table 15–10. In the second step, the rules are evaluated. The use of the γ operator with $\gamma = 0$ is equivalent to the multiplication of the degree of membership of the condition and the degree of sensibleness. The results are summarized in table 15–11, where the maxima of the respective degree of membership for the two terms (urgent, not_urgent) of the linguistic variable "date criteria" are printed in bold. Part 3 in the table shows the highest degree of membership in the fuzzy set of parts with urgent date criteria and the lowest degree of membership in the fuzzy set of parts with not_urgent date criteria.

Results. The approach described above has been programmed, and its performance has been compared to systems with no master scheduling and employing

Table 15–9. Membership grades for slack time and waiting time.

		Membership grade of part		
		1	2	*3*
Waiting time:	long	0.7	0	0.7
	medium	0.2	0.8	0.3
	short	0	0.4	0
Slack time:	critically_short	0.4	0.8	0.7
	short	0.6	0.2	0.3

Table 15–10. Membership grades for conditional parts of the rules.

	Part		
	1	*2*	*3*
Condition 1	0.58	0.00	0.67
Condition 2	0.20	0.78	0.41
Condition 3	0.00	0.53	0.00
Condition 4	0.72	0.00	0.41
Condition 5	0.29	0.37	0.21
Condition 6	0.29	0.37	0.21

Table 15–11. Membership grades for the rules.

	Part		
	1	*2*	*3*
Date criterion is urgent:			
conclusion 1	**0.58**	0.00	**0.67**
conclusion 2	0.16	**0.62**	0.33
conclusion 3	0.00	0.32	0.00
conclusion 4	0.36	0.00	0.21
conclusion 5	0.06	0.07	0.04
Date criterion is not urgent:			
conclusion 6	**0.20**	**0.26**	**0.15**

only simple priority rules for release and machine scheduling using a general simulation program for FMS. The results are shown in table 15–12. The suggested approach dominated the classical priority scheduling with respect to all three objectives.

15.3.3.3 Aggregate Production and Inventory Planning [Rinks 1982a, b]. The "HMMS-model" [Holt et al. 1960] is one of the best-known classical models in aggregate production planning. It assumes that the main objective of the production planner is to minimize total cost, which is assumed to consist of costs of regular payroll, overtime and layoffs, inventory, stock-outs, and machine setup. The model assumes quadratic cost functions and then derives linear decision rules for the production level and the work-force level. The following terminology is used:

Table 15–12. Results.

Criteria	*Suggested approach*	*Priority rule approach*
Mean in-process waiting time [min]	2,884	3,369
Part of lots that have met their due dates [%]	97	28
Mean machine utilization [%]	80	79

FS_t = sales forecast for period t
W_{t-1} = work force level in period $t - 1$
I_{t-1} = inventory level at the end of period $t - 1$
ΔW_t = change in work force level in period t
P_t = production level in period t

In general, the decision variables are related to the cue variables as

$$P_t = f(FS_t, W_{t-1}, I_{t-1})$$
$$\Delta W = g(FS_t, I_{t-1})$$

By contrast to most other models, the HMMS-model was tested empirically for a paint factory. The cost coefficients were derived in different ways (statistically, heuristically, etc.), and the performance of the decision rules was compared to the actual performance of the paint factory managers [Holt et al. 1960].

The following model resulted for the paint factory.

Model 15–5

$$\text{minimize } C_N = \text{minimize} \sum_{t=1}^{V} C_t$$

where

$$\begin{aligned} C_t = {} & [340 W_t] && \text{Regular payroll costs} \\ & + \left[64.3(W_t - W_{t-1})^2\right] && \text{Hiring and layoff costs} \\ & + \left[0.20(P_t - 5.67W_t)^2 + 51.2P_t - 281W_t\right] && \text{Overtime costs} \\ & + \left[0.0825(I_t - 320)^2\right] && \text{Inventory-connected costs} \end{aligned}$$

and subject to restraints

$$I_{t-1} + P_t - S_t = I_t \quad t = 1, 2, \ldots, N$$

Even though the HMMS-model performed quite well and is used as a common benchmark for later models, it was rarely used in practice. The main objection was generally that managers would not use it, roughly speaking, because too much mathematics was involved.

Rinks tries to avoid this lack of acceptance by suggesting a model based on the concepts described in chapters 9 and 10 of this book. He developed one production and one work-force algorithm that consist of a series of relational assignment statements (rules) of the form

If FS_t is . . . and I_{t-1} is . . .
and W_{t-1} is . . . then P_t is . . .
Else . . .

and

If FS_t is . . . and I_{t-1} is . . .
and W_{t-1} is . . . then ΔW_t is . . .
Else . . .

respectively

He uses the definition (given in table 15–13) of the terms of linguistic variables. Figure 15–22 sketches the membership functions of the terms of the linguistic variables used. Forty decision rules were suggested (see table 15–14), these were not claimed to be optimal but rather heuristic in character and acceptable to the manager.

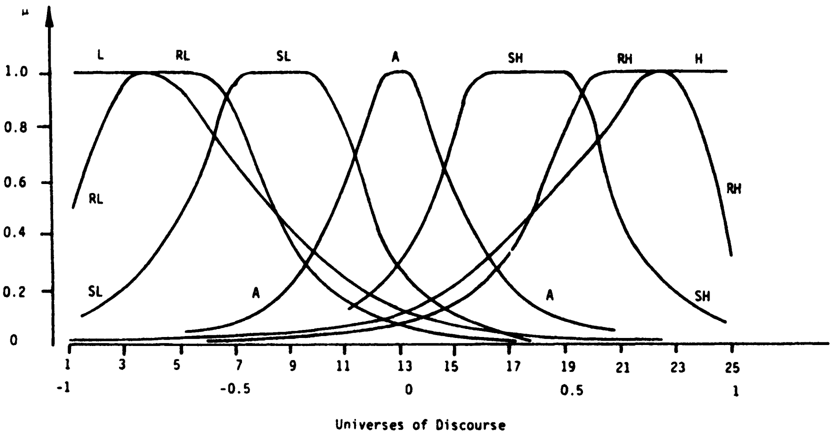


Figure 15–22. Membership functions for several linguistic terms.

Table 15–13. Definition of linguistic variables [Rinks 1982].

Linguistic terms	*Acronym*	*Base variable*[a]	*Membership function expression*[b,c]
VERY HIGH (POSITIVE, VERY BIG)	VH (PVB)	x (dx)	HIGH x * HIGH x
HIGH (POSITIVE BIG)	H (PB)	x (dx)	$1 - e^{[-(0.5/\|1-x\|)^{2.5}]}$
RATHER HIGH (POSITIVE, RATHER BIG)	RH (PRB)	x (dx)	$1 - e^{[-(0.25/\|0.7-x\|)^{2.5}]}$
SORTOF HIGH (POSITIVE, SORTOF BIG)	SH (PSB)	x (dx)	$1 - e^{[-(0.25/\|0.4-x\|^{2.5}]}$
AVERAGE (ZERO)	A (Z)	x (dx)	$1 - e^{[-5\|x\|]}$
SORTOF LOW (NEGATIVE, SORTOF BIG)	SL (NSB)	x (dx)	$1 - e^{[-(0.25/\|-0.4-x\|)^{2.5}]}$
RATHER LOW (NEGATIVE, RATHER BIG)	RL (NRB)	x (dx)	$1 - e^{[-(0.25/\|-0.7-x\|)^{2.5}]}$
LOW (NEGATIVE BIG)	L (NB)	x (dx)	$1 - e^{[-(0.5/\|1-x\|)^{2.5}]}$
VERY LOW (NEGATIVE, VERY BIG)	VL (NVB)	x (dx)	LOW x * LOW x
AT LEAST AVERAGE	ALA	x	$1 - e^{[-5\|x\|]}$ $\quad -1 \leqslant x \leqslant 0$ 1 $\quad 0 < x \leqslant 1$
AT MOST AVERAGE	AMA	x	1 $\quad -1 \leqslant x \leqslant 0$ $1 - e^{[-5\|x\|]}$ $\quad 0 < x \leqslant 1$

[a] x is any one of the following variables: W_{t-1}, FS_t, W_t, and P_t. dx is ΔW_t.
[b] All variables are scaled to be placed in the [−1, 1] interval.
[c] dx replaces x in the membership function expression for use with ΔW_t.

To test the performance of the suggested approach, the data of the paint factory of the HMMS-model were used. In order to apply Rinks decision rules, the membership functions of the terms, as shown in figure 15–22, had to be calibrated. In fact the range [−1, 1] on the horizontal axis of this figure had to be calibrated to the data. For test purposes, lower and upper bounds as shown in the following tabulation were derived from available historical data (HMMS):

Table 15–14. Membership functions.

	Cue variables			Decision variables	
Rule no.	FS_t	I_{t-1}	W_{t-1}	P_t	ΔW_t
1	H	AMA	H	H	Z
2	H	AMA	A	RH	PRB
3	H	AMA	L	SH	PVB
4	SH	L	H	H	Z
5	SH	L	A	RH	PRB
6	SH	L	L	SH	PVB
7	SH	SH	H	SH	NRB
8	SH	SH	A	A	Z
9	SH	SH	L	A	PRB
10	A	A	H	SH	NRB
11	A	A	A	A	Z
12	A	A	L	A	PRB
13	SL	SL	H	SH	NRB
14	SL	SL	A	A	Z
15	SL	SL	L	SL	PRB
16	RL	L	H	SH	NRB
17	RL	L	A	A	Z
18	RL	L	L	A	PRB
19	L	ALA	H	SL	NVB
20	L	ALA	A	RL	NRB
21	L	ALA	L	L	Z
22	SL	H	H	SL	NVB
23	SL	H	A	RL	NRB
24	SL	H	L	RL	Z
25	H	AMA	SH	H	PSB
26	H	AMA	SL	SH	PB
27	SH	L	SH	H	PSB
28	SH	L	SL	SH	PB
29	SH	SH	SH	A	Z
30	SH	SH	SL	A	PSB
31	A	A	SH	A	NSB
32	A	A	SL	A	PSB
33	SL	SL	SH	A	NSB
34	SL	SL	SL	A	Z
35	RL	L	SH	A	NSB
36	RL	L	SL	A	Z
37	L	ALA	SH	RL	NB
38	L	ALA	SL	L	NSB
39	SL	H	SH	RL	NB
40	SL	H	SL	RL	Z

1. Acronyms for the values of the linguistic variables are defined in table 15–13.
2. Each production rule is a fuzzy relational assignment statement of the form "IF FS_t is ______ AND I_{t-1} is ______ AND W_{t-1} is ______ THEN P_t is ______."
3. Each work force rule is a fuzzy relational assignment statement of the form "IF FS_t is ______ AND I_{t-1} is ______ AND W_{t-1} is ______ THEN ΔW_t is ______."

Variable	*Lower bound*	*Upper bound*
W_{t-1}	60	115
ΔW_t	−10	10
P_t	250	750
I_{t-1}	150	490
FS_t	250	750

In the absence of historical data, the manager would use his or her judgment to make the determinations. For computations, the max-min compositions were used, resulting in fuzzy sets as representing the "conclusion" or "decision." Since, however, a decision concerning the workforce, production, or inventory of next period should be a crisp decision, Rinks used the maximum rule if possible. If the membership function did not have a unique maximum he used other, heuristic rules to choose the crisp decision to be implemented.

For the 60 months of data for the HMMS-model (1949–1953), the results of the work-force algorithm are shown in figure 15–23. The cost results are shown in table 15–15.

Rink's own evaluation of the simulation results reads as follows:

> While the 5.0 per cent cost penalty evidenced by the production scheduling fuzzy algorithms is somewhat greater than that reported by other heuristics—Search Decision

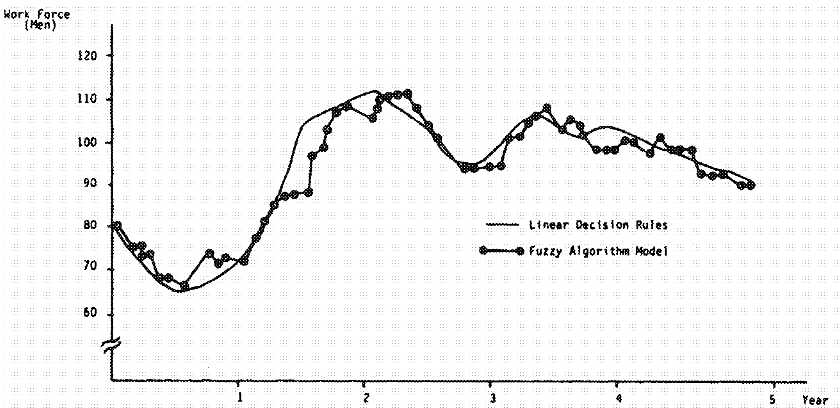


Figure 15–23. Comparison of work force algorithms.

Table 15–15. Cost results.

Costs (1,000\$)	*Linear DR HMMS (optimal)*	*Fuzzy algorithm*
Regular payroll	1,879	1,814
Hiring and layoff	20	22
Overtime	129	251
Inventory	25	43
Total cost	2,053	2,130

Table 15–16. Comparison of performances.

	Ice cream	*Chocolate*	*Candy*	*Paint*
Decision rule (perfect)	100%	100%	100%	100%
Decision rule (moving average)	104.9%	102.0%	103.3%	110%
Company performance	105.3%	105.3%	114.4%	139.5%
Management coefficients	102.3%	100.0%	124.1%	124.7%
Correlation	$W^{t,r} = .78$ $P^{t,r} = .97$	$W^{t,r} = .57$ $P^{t,r} = .93$	$W^{t,r} = .73$ $P^{t,r} = .86$	$W^{t,r} = .40$ $P^{t,r} = .66$

> Rule [Taubert 1967] and Parametric Production Planning [Jones 1967] reported cost penalties of less than one percent for the paint factory—it must be ramembered that the fuzzy algorithms do not even require an explicit cost function. For situations where restrictive assumptions cannot be retionalized and sufficient data is not available to construct a cost function, approximate reasoning based models would seem to offer an appealing alternative [Rinks 1982b, p. 579].

If Rinks had compared his results to other benchmarks, he would probably have been more optimistic. Table 15–16 is from Bowman [1963, p. 104] and shows the real performance and the performance of another heuristic, the management coefficient approach, in the case of the HMMS paint factory and three other plants. Compared to the 139% and 124.7% performance of these two approaches, the 105% performance of the fuzzy algorithm would look even better.

15.3.3.4 Fuzzy Mathematical Programming for Maintenance Scheduling. The following application, basing on a master thesis from Zittau, Germany, is of interest because the effects of different operators were investigated and because parametrized membership functions were used.

Model 15–6 [Holtz and Desonki 1981]

The problem objective here is to determine optimal maintenance cycles in electrical power plants. Stochastic models had been used before, but because of the very low frequency of breakdowns, it seemed that a model based on frequentistic arguments was not appropriate.

T_j : Cycle times of maintenance operations for $j = 1, \ldots, N$ maintenance crews (decision variable)
x_{ij} : Coefficients of the crisp cost function, $i = 1, 2, 3; j = 1, \ldots, N$
y_{ij} : Coefficients of the manpower requirement function, $i = 1, 2, 3; j = 1, \ldots, N$
z_{ij} : Coefficients of the breakdown function, $i = 1, 2, 3; j = 1, \ldots, N$
Mh : Number of manhours available for maintenance per year
B : Number of breakdowns per year
$B_{\max}$: Maximum of acceptable breakdowns per year

Crisp Mathematical Model. For $N = 2$ and C = total cost, the following crisp model was the point of departure:

$$\text{minimize} \quad C = C(T_1, \ldots, T_n) = \sum_{j=1}^{n} \left(x_{1j}T_j + x_{2j} + \frac{x_{3j}}{T_j} \right)$$

$$\text{such that} \quad \sum_{j=1}^{N} \left(y_{1j}T_j + y_{2j} + \frac{y_{3j}}{T_j} \right) \leq Mh$$

$$\sum_{j=1}^{N} \left(z_{1j}T_j + z_{2j} + \frac{z_{3j}}{T_j} \right) \leq B_{\max}$$

$$T_j \geq 0$$

The requirements were as follows:

1. Cost should not exceed 500 considerably—and in no case should exceed an upper bound that could be varied.
2. Manpower Mh should generally not exceed 1,100, and by no means 1,200.
3. The number of breakdowns can exceed 50 but never 300 ($B_{\max}$).

Fuzzy Mathematical Model. The symmetrical concept of a decision (definition 14–1) was used, and the optimal decision was defined to be

$$T_{j0} = T_j \circ \mu_{\tilde{T}_j} = \max_j \min_i \mu_i(T_j)$$

Two types of membership functions were investigated: a linear membership function and a nonlinear two-parameter membership function.

Type 1 Membership Functions

$$\mu_{\tilde{C}}(T_j) = \frac{1}{2}\left\{[1 + \operatorname{sgn}(C_L - C)] + [1 + \operatorname{sgn}(C - C_L)] \cdot \left(\frac{C_U - C}{C_L - C_U}\right)\right\}$$

where C_L and C_U represent the lower and upper bounds for total cost.

$$\mu_{\tilde{M}h}(T_j) = \frac{1}{2}\left\{[1 + \operatorname{sgn}(Mh_L - Mh)] + [1 + \operatorname{sgn}(Mh - Mh_L)] \cdot \left(\frac{Mh_U - Mh}{Mh_L - Mh_U}\right)\right\}$$

with Mh_L and Mh_U the lower and upper bounds.

$$\mu_{\tilde{B}}(T_j) = \frac{1}{2}\left\{[1 + \operatorname{sgn}(B_L - B)] + [1 + \operatorname{sgn}(B_U - B_L)] \cdot \left(\frac{B_U - B}{B_L - B_U}\right)\right\}$$

with B_L and B_U the lower and upper bounds.

Type 2 Membership Function. We shall only show the membership function for the objective function. The others are defined accordingly:

$$\mu_{\tilde{C}}(T_j) = \frac{1}{2}\left\{[1 + \operatorname{sgn}(C_L - C)] + \frac{1 + \operatorname{sgn}(C - C_L)}{1 + \left(\frac{1}{b_1} - 1\right)\left|\frac{C - C_L}{c_1}\right|}\right\}$$

b_1 and c_l serve as means of better fitting the membership function to the real situation. On the other hand, they obviously increase the computational effort.

Detailed numerical results, as well as a comparison of the performance of the min-operator versus the product operator as a model for the intersection, can be found in Holtz [1981].

15.3.3.5 Scheduling Courses, Instructors, and Classrooms. It is well known that the determination of time schedules in which several resources have to be combined belongs to the most difficult combinatorial problems in operations research. Rarely does one ever try to determine optimal schedules. The determination of feasible schedules is very often the best one can hope for. The difficulty of obtaining such schedules by formal algorithms might partly be due to the fact that constraints are treated as crisp requirements even though in reality they often are flexible. The following case indicates how a combination of fuzzy set theory and heuristics can lead to quite acceptable results.

Model 15–7 [Prade 1979]

Problem Description. A quarterly schedule in a French university is to be determined. There are N (here $N = 4$) instruction programs; each lasts one year, and a student can only attend one of them. Each instruction program I consists of $M(I)$ courses (here, $10 \leq M(I) \leq 14$). Each course contains lectures, lab work, and a final examination.

A course is taught by one instructor, supported by several teaching assistants. An instructor may teach several courses in one or several instruction programs. The availability of an instructor differs from person to person. An instructor may be present for only some predetermined days of a week; another may be available for only some weeks during the quarter. Information about the availability of instructors is only known approximately beforehand.

A schedule has to satisfy seven "global" constraints:

1. Each instruction program must be completely planned for the entire school year.
2. There are precedence constraints between courses (or sometimes parts of courses) that are elements of the same instruction program.
3. It is not desirable that more than four weeks elapse between the first lecture of a course and its final examination.
4. It is not desirable that any course that has already begun is interrupted for more than a week.
5. Some courses can be in common in several instruction programs.
6. An instructor is not always available.
7. It is very desirable that several courses (three or four) are planned during the course of the same week.

Constraints 1, 2, 5, and 6 are considered as "hard," 3, 4, and 7 as "soft" constraints. More local constraints will be considered later.

Solution. The flow time of a course is considered as a fuzzy number with a membership function similar to that shown in figure 15–24. These fuzzy numbers in L-R representation (see definition 5–6) are used to compute via fuzzy PERT a fuzzy early starting date $\tilde{r}_i$ and a late ending date $\tilde{d}_i$. If x denotes time, then the

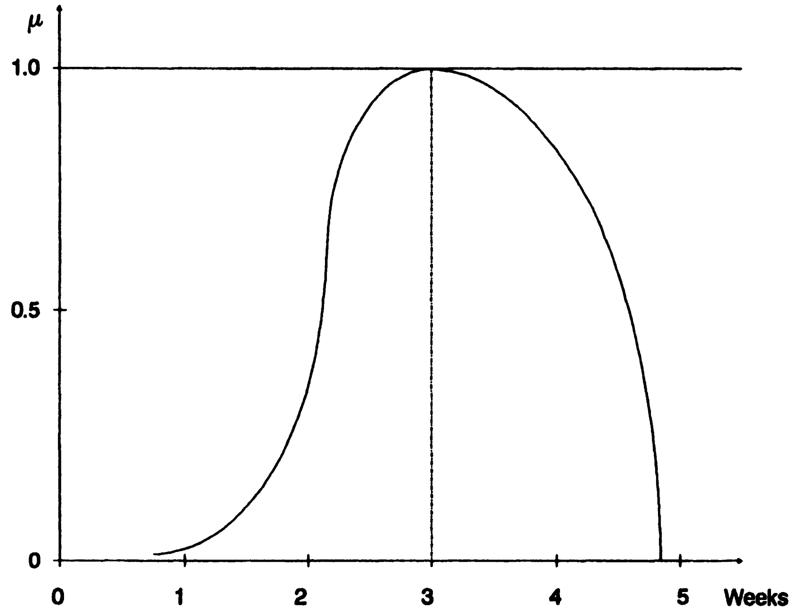


Figure 15–24. Flowtime of a course.

interval $\tilde{I}_i$ in which the course i will be taught is a fuzzily bounded interval (see figure 7–5), bounded by $\tilde{r}_i$ and $\tilde{d}_i$, respectively.

The membership function of these intervals $\tilde{I}_i$ is

$$\mu_{\tilde{I}_i}(x) = \begin{cases} \mu_{\tilde{r}_i}(x) & \text{for} \quad x \le r_i \\ 1 & \text{for} \quad x \in [r_i, d_i] \\ \mu_{\tilde{d}_i}(x) & \text{for} \quad x \ge d_i \end{cases}$$

where r_i, d_i are the mean values of $\tilde{r}_i$, and $\tilde{d}_i$, respectively.

The "global" constraints are taken into consideration successively: Constraints 1 and 2 are used as a basis for PERT; constraint 4 is used to compute whole programs from single courses. And if constraint 5 is relevant, the intersection of the different possibility intervals for all relevant courses in all effected instruction programs is computed. Constraint 6 is taken care of similarly.

So far, the slack time for each course, the work load of each instructor, and the number of courses per week for each instruction program have been determined. Modifications of this schedule due to the availability of the instructors

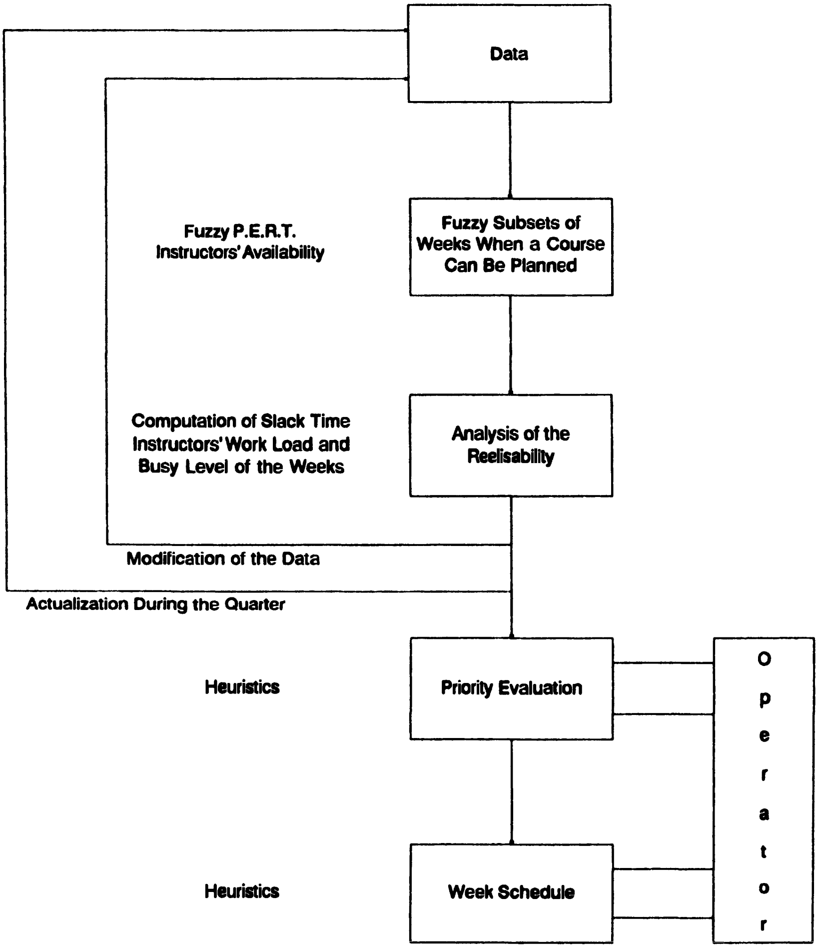


Figure 15–25. The scheduling process.

can now be made, and the following "local" constraints are considered by interactively changing schedules that have been generated automatically via heuristic priority assignment. Figure 15–25 summarizes the entire process.

"Local" constraints are as follows:

1. There exist precedence constraints between lectures and lab work inside a course (the graph of these constraints is not the same for all the courses).
2. An instructor can teach only one lecture at a given moment.

3. It is generally desirable to plan two lectures of the same course in succession, but not three.
4. It is not desirable that an instructor teach more than two lectures of different courses in the same morning.
5. It is desirable to give priority to lectures in the morning and lab work in the afternoon.

Example 15–4

The following tables and figures can only serve to visualize the process. Details can be found in Prade [1977]. Figure 15–26 presents the data of one of the four instruction programs that were considered. All courses had to be scheduled within one quarter of 11 weeks. Table 15–17 gives the node number, name of courses, instructor number, and category (1 to 4 indicate different availabilities of the instructor). p, α, and β are the mean values of the left and right spreads of the processing time for each course. The availability for each instructor is given in table 15–18. Table 15–19 gives course numbers, initialized by the name of the instructor and early start and late finish times.

Table 15–17. Structure of instruction program.

		Instructor category number		*Processing time*		
N	*Name*			α	p	β
1	A231 (L1 to 6)	9	3	0.5	2	1
2	A231 (L7 to 10)	9	3	0.5	1	1
3	A141 (L1 to 6)	12	3	0.5	2	1
4	A141 (L7 to 10)	12	3	0.5	1	1
5	A121	12	3	1	3	1.5
6	A241	12	3	1	3	1.5
7	A510	9	3	1	3	1.5
8	M317 (L1 to 8)	17	4	1	3	1.5
9	M317 (L9 to 10)	17	4	1	3	1.5
10	PS16	8	1	0	4	0
11	V231	21	2	0	1	0
12	V211	1	4	1	3	1.5
13	E541	23	4	1	3	1.5
14	E551	23	4	1	3	1.5
15	M361	13	3	1	3	1.5
16	E531	11	1	0	4	0
17	E532	11	1	0	4	0

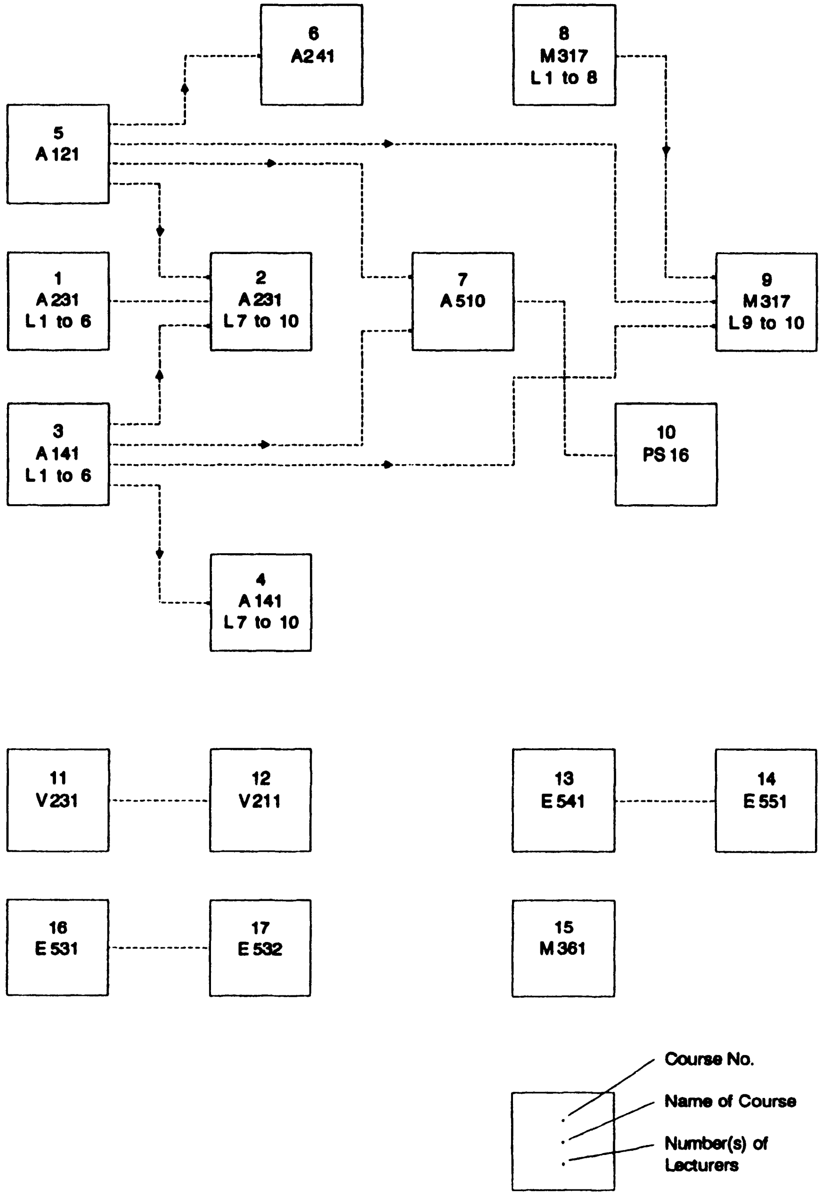


Figure 15–26. Courses of one instruction program.

Table 15–18. Availability of instructors.

Instructor number	Weeks 1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0.5	1	1	1	1	1	0.5
8	0	0	0	0	0.5	1	1	1	1	0.5	0.5
9	1	1	1	1	1	1	1	1	1	0	1
11	0	0	0	0.5	1	1	1	1	1	1	1
12	1	1	0.5	0	1	1	1	1	1	1	1
13	1	1	1	0.5	0	1	1	1	1	0.5	0.5
17	1	1	1	1	1	1	0.5	0	0.5	1	1
21	1	1	0.5	0	0	0	0	0	0	0	0
23	1	1	1	1	1	1	0.5	0.5	0.5	0	0

Table 15–19. PERT output.

Name	α	$\bar{p}$	β	α	d	β
A231	0	1	0	0	11	0
A141	0	1	0	2	6	2
A121	0	1	0	1.5	4	1
A241	1	4	1.5	0	11	0
A510	1	4	1.5	0	7	0
M317	0	1	0	0	11	0
PS16	2	7	3	0	11	0
V231	0	1	0	0	3	0
V211	0	5	0	0	11	0
E541	0	1	0	1.5	6	1
E551	1	4	1.5	0	9	0
M361	0	1	0	0	11	0
E531	0	4	0	0	7	0
E532	0	8	0	0	11	0

As reference (membership) functions for the fuzzy numbers in L-R-representation representing the "flowlines" of the course, Prade used $L(x) = \exp[-x^2]$ and $R(x) = \max[0, 1 - x^2]$.

The intersection of the availability schedule (table 15–18) and the PERT schedule yields the possibility schedule of weeks in which courses can be scheduled (table 15–20). Table 15–21 shows an example of the final schedule for the first week.

Table 15–20. Availability of weeks for courses.

	Weeks										
Name	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
A231	1	1	1	1	1	1	1	1	0	1	1
A141	1	1	0.5	0	1	1	0.8	0.4	0	0	0
A121	1	1	0.5	0	0.4	0	0	0	0	0	0
A241	0	0	0.4	0	1	1	1	1	1	1	1
M510	0	0	0.4	1	1	1	1	0	0	0	0
M317	1	1	1	1	1	1	0.5	0	0.5	1	1
PS16	0	0	0	0	0.4	0.8	1	1	1	0.5	0.5
V231	1	1	0.5	0	0	0	0	0	0	0	0
V211	0	0	0	0	0.5	1	1	1	1	1	0.5
E541	1	1	1	1	1	1	0.4	0	0	0	0
E551	0	0	0.4	1	1	1	0.5	0.5	0.5	0	0
M361	1	1	1	0.5	0	1	1	1	1	0.5	0.5
E531	0	0	0	0.5	1	1	1	0	0	0	0
E532	0	0	0	0	0	0	0	1	1	1	1

Table 15–21. First week's final schedule.

	Morning			*Afternoon*
Monday	A141 L.1	A141 L.2	A121 L.1	—
Tuesday	A231 L.1	A231 L.2	M317 L.1	—
Wednesday	A231 L.3	A231 L.4	M317 L.2	A231 L.W.1
Thursday	A141	A141	A121	Sports
Friday	A121	A121	M317	A141

15.3.4 Fuzzy Models in Inventory Control

There exist a large number of inventory models in operations research using a great variety of methods for their solution. For inventory models using linear or integer linear models, the approach of section 14.2 or an algorithm described in Zimmermann and Pollatschek [1984] may be used. For solutions basing on differential calculus, the models in chapter 7 might be useful. Kacprzyk and Staniewksi [1982] present a very interesting approach for aggregate inventory planning, using primarily the concept presented in chapters 3 and 5 of this book. We shall present a model that uses Bellman and Zadeh's approach to fuzzy dynamic programming discussed in section 4.3.

Model 15–8 [Sommer 1981]

The management of a company wants to close down a certain plant within a definite time interval. Therefore production levels should decrease to zero as steadily as possible and the stock level at the end of the planning horizon should be as low as possible. The demand is assumed to be deterministic.

Mathematical model. Let

$d_i \in D, i = 1, \ldots, N$ be the decision variable representing the production level in period i,

where

$D = \{\alpha_1, \ldots, \alpha_n\}$ is the set of values permitted for the decisions.
$\tilde{x}_i \in X, i = 1, \ldots, N+1$ be the state variable representing the inventory level at the beginning of period i,

and

$X = \{\tau_1, \ldots, \tau_m\}$ is the set of possible state values,
$a_{i,i} = 1, \ldots, N$ is the deterministic demand in period i,
$x_{i+1} = x_i + d_i - a_i$ is the crisp transformation function,
$\tilde{C}_i(d_i) = \{(d_i, \mu_{\tilde{C}_i}(d_i)\}$ are fuzzy constraints on the decision variables representing the goal "production should decrease as steadily as possible,"
$i = 1, \ldots, N$, and
$\tilde{G}_i(x_{N+1}) = \{x_{N+1}, \mu_{\tilde{G}}(x_{N+1})\}$ is the fuzzy goal, representing the decision to have as low a stock level as possible at the end of the planning horizon.

Then, using equation (14.20), the membership function of the decision on stage i is

$$\mu_{\tilde{D}}(d_i) = \min\{\mu_{\tilde{C}_i}(d_i), \mu_{\tilde{G}}(x_{N+1})\}$$

and the membership function of the maximizing decision on stage i is

$$\mu_{\tilde{D}}^0(d_i) = \max_{d_i \in D}\{\min\{\mu_{\tilde{C}_i}(d_i, \mu_{\tilde{G}}(x_{N+1})\}\}$$

which can be determined recursively using equation (14.23), As will be shown in the following numerical example, the state spaces can sometimes be reduced even further by introducing a bound on the basis of heuristic considerations.

Example 15–5

Let

$$\mu_{\tilde{C}_i}(d_i) = \begin{cases} 0 & \text{if} \quad 0 \leq d_i \leq 60 - 10i \\ -3 + .5i + d_i/20 & \text{if} \quad 60 - 10i \leq d_i \leq 80 - 10i \\ 5 - .5i - d_i/20 & \text{if} \quad 80 - 10i \leq d_i \leq 100 - 10i \\ 0 & \text{if} \quad 100 - 10i \leq d_i \end{cases}$$

and

$$\mu_{\tilde{G}_{N+1}}(x_{N+1}) = \begin{cases} 1 - x_{N+1}/20 & \text{if} \quad 0 \leq x_{N+1} \leq 20 \\ 0 & \text{else} \end{cases}$$

$$a_1 = 45, \quad a_2 = 50, \quad a_3 = 45, \quad a_4 = 60, \quad \text{and} \quad N = 4$$

x, the stock level at the beginning, is supposed to be 0.

$$\tau_j = \{0, 5, 10, \ldots\}$$
$$\alpha_h = \{0, 5, 10, \ldots\}$$

Only $\{d_i \mid \mu_{\tilde{C}_i}(d_i) > 0\}$ are of interest. Hence we can put a bound on the decision variables as follows:

i	d_i^l	d_i^u
1	55	85
2	45	75
3	35	65
4	25	55

Also, $0 < x_5 \leq 20$.

Using the transformation function, we can also find upper and lower bounds for the state variables on the different intermediate stages. We proceed in three steps: First we determine upper bounds $x_i^{u'}$ and lower bounds $x_i^{l'}$ from the forward calculation. The according bounds $x_i^{u''}$ and $x_i^{l''}$ from backward calculation are computed in the second step. Then we can obtain the final bounds by

$$x_i^u = \min\{x_i^{u'}, x_i^{u''}\}$$
$$x_i^l = \max\{x_i^{l'}, x_i^{l''}\}$$

The lower bound for the state variable x_i can be calculated as

$$x_i^{l'} = \max\{0, x_{i-1}^{l'} + d_{i-1}^l - a_{i-1}\} \quad i = 2, \ldots, 4$$

The appropriate upper bound is

$$x_i^{u'} = x_{i-1}^{u'} + d_{i-1}^u - a_{i-1} \quad i = 2, \ldots, 4$$

For the different stages we obtain, for $x_1 = 0$,

i	$x_i^{l'}$	$x_i^{u'}$
1	—	—
2	10	40
3	5	65
4	0	85
5	—	—

Starting with x_5 and assuming $x_5^{1''} = 0$ and $x_5^{4''} = 20$, we obtain recursively the following upper and lower bounds:

i	$x_i^{l''}$	$x_i^{u''}$
1	—	—
2	0	65
3	0	60
4	5	50
5	—	—

The final upper and lower bounds can be determined by

$$x_i^l = \max\{x_i^{l'}, x_i^{l''}\}$$
$$x_i^u = \min\{x_i^{u'}, x_i^{u''}\}$$

Hence

i	x_i^l	x_i^u
1	0	0
2	10	40
3	5	60
4	5	50
5	0	15

Now we can determine the optimal d_i and x_i within the lower and upper bounds computed above:

Stage 1: Using equation (14.23), we obtain

$$\mu_{\tilde{G}_4}(x_4) = \max_{d_4}\{\min[\mu_{\tilde{C}}(d_4), \mu_{\tilde{G}}(x_4, d_4)]\}$$
$$= \max_{d_4}\{\min[\mu_{\tilde{G}}(d_4), \mu_{\tilde{G}}(x_4 + d_4 - a_4)]\}$$

	d_4							
x_4	25	30	35	40	45	50	55	$\mu_{\tilde{G}_4}(x_4)$
5							1/4	1/4
10						1/2	1/4	1/2
15					3/4	1/2	1/4	3/4
20				1	3/4	1/2	1/4	1
25			3/4	3/4	1/2	1/4		3/4
30		1/2	3/4	1/2	1/4			3/4
35	1/4	1/2	1/2	1/4				1/2
40	1/4	1/2	1/4					1/2
45	1/4	1/4						1/4
50	1/4							1/4

Stage 2: $\mu_{\tilde{D}}(x_3) = \max_{d_3}\{\min[\mu_{\tilde{C}}(d_3), \mu_{\tilde{G}}(x_3 + d_3 - a_3)]\}$

x_3	d_3 35	40	45	50	55	60	65	$\mu_{\tilde{G}_3}(x_3)$
5			1/4	1/2	3/4	1/2	1/4	3/4
10		1/4	1/2	3/4	3/4	1/2	1/4	3/4
15	1/4	1/2	3/4	1	3/4	1/2	1/4	1
20	1/4	1/2	3/4	3/4	3/4	1/2	1/4	3/4
25	1/4	1/2	3/4	3/4	1/2	1/2	1/4	3/4
30	1/4	1/2	3/4	1/2	1/2	1/4	1/4	3/4
35	1/4	1/2	1/2	1/2	1/4	1/4		1/2
40	1/4	1/2	1/2	1/4	1/4			1/2
45	1/4	1/2	1/4	1/4				1/2
50	1/4	1/4	1/4					1/4
55	1/4	1/4						1/4
60	1/4							1/4

Stage 3: $\mu_{\tilde{D}}(x_2) = \max_{d_2}\{\min[\mu_{\tilde{C}}(d_2), \mu_{\tilde{D}}(x_2 + d_2 - a_2)]\}$

x_2	d_2 45	50	55	60	65	70	75	$\mu_{\tilde{G}_2}(x_2)$
10	1/4	1/2	3/4	3/4	3/4	1/2	1/4	3/4
15	1/4	1/2	3/4	3/4	3/4	1/2	1/4	3/4
20	1/4	1/2	3/4	3/4	1/2	1/2	1/4	3/4
25	1/4	1/2	3/4	1/2	1/2	1/2	1/4	3/4
30	1/4	1/2	1/2	1/2	1/2	1/4	1/4	1/2
35	1/4	1/2	1/2	1/2	1/4	1/4	1/4	1/2
40	1/4	1/2	1/2	1/4	1/4	1/4		1/2

Stage 4: $\mu_{\tilde{D}}(x_1) = \max_{d_1}\{\min[\mu_{\tilde{C}}(x_1), \mu_{\tilde{D}}(x_1 + d_1 - a_1)]\}$

x_1	d_1 55	60	65	70	75	80	85
0	1/4	1/2	3/4	3/4	1/2	1/2	1/4

15.3.5 Fuzzy Sets in Marketing

Classical applications of fuzzy sets in marketing, such as media selection [Wiedey and Zimmermann 1978], are too similar to other applications of fuzzy linear programming to be discussed here again. We shall rather turn to recent and modern applications of fuzzy technology in marketing, which base primarily on data mining. The motivation and justification of these applications is, of course, the change of data availability that has already been mentioned before.

The so-called database marketing became only feasible after enough data were available. Unluckily the data masses grew that fast, that they superceded quickly the human competence of percepting complex and little structured data pools.

In the following we shall focus on the area of market segmentation and present first an easy to understand static problem and then a more complicated dynamic problem.

15.3.5.1 Customer Segmentation in Banking and Finance. Banks and insurance companies have now-a-days masses of data about customers stored in their data banks and data warehouses. They have a number of products that they want to offer their customers, such as shares, bonds, derivates, etc. and they develop new products. On the other hand, each individual customer has certain wishes and needs, preferring one or the other of the products, i.e. they have certain product affinities. If a bank would offer all its products to all its customers (for instance, by a mailing) then it would be very costly, the relative effectiveness would be very low and the customers might even be frustrated by getting that much mail.

In databank marketing the bank tries to subdivide its customers into segments which are as homogenous with respect to the needs of the customers in a segment. One can then offer special products only to segments which have a high demand for this product. This is called customer segmentation.

Traditionally the features of these segments are defined as crisp intervals concerning, e.g. property, debt, income, balance of account, age, etc. and nominal features, such as sex, marital status, profession, etc.

The main disadvantages of these feature definitions are, that there is no compensation between the features, that wrong classifications occurs and that dynamic changes of the customers cannot be accounted for. If fuzzy analysis, e.g. fuzzy clustering, is used, marginal customers are better classified, existing compensations can be considered and dynamic changes may be recognized via changes of degrees of membership of customers to clustery.

Example 15–6

In this experimental study of 300 customers the following features were used:

Income
Credit
Age
Property
Profit (of the bank).

Traditionally the bank distinguished three classes:

Class 1:	Annual income less than	DM 30,000
	Property less than	DM 40,000
Class 2:	Income between	DM 30,000 and 80,000
	Property between	DM 40,000 and 200,000
Class 3:	Income more than	DM 80,000
	Property more than	DM 200,000

The marketing effect turned out to be very unsatisfactory.

Subsequently fuzzy clustering was used. The fuzzy c-means algorithm, described in chapter 13 was used. It turned out, that nine classes represented the optimal number of classes and shown in table 15–22 class centers were determined:

These classes would certainly not have been found by a traditional classification. At the first glance those classes do not really make much sense. When shown to marketing experts, however, they found the following very plausible description of the classes:

Class:	*Content:*
1	more than 60-year-old persons with low income and a certain property
2	in training
3	3rd stage of life high income and property
4	3rd stage of life low income and property
5	career persons
6	high senior citizens' segment
7	junior segment
8	persons with high credit capacity
9	social weak segment

15.3.5.2 Bank Customer Segmentation based on Customer Behavior [Angstenberger 2001]. By contrast to last section this time not the present status of the customers (snap shot like) but the dynamic behavior of the customers will be used for segmentation. The study, which is described here, is much larger than that in last section. It concerns 24,267 customers of a commercial bank and it is described in detail in [Angstenberger 2001]. Here we summarize the problem, the process and the results.

Table 15–22. Cluster centers of nine optimal classes.

	Age	*Income*	*Property*	*Credit*	*Profit*
class 1	63	2,585	10,485	2,965	45
class 2	28	2,300	8,020	3,200	24
class 3	52	5,260	50,920	7,830	256
class 4	53	3,200	20,785	6,040	165
class 5	43	6,240	22,680	8,925	117
class 6	78	2,190	34,280	1,185	316
class 7	9	235	3,285	230	5
class 8	42	1,120	15,705	150,060	930
class 9	37	955	13,405	7,302	0

Table 15–23. Dynamic features describing bank customers.

Feature	*Description*
1	Overdraw limit on account
2	Current end-of-month balance
3	Maximum balance this month
4	Minimum balance this month
5	Average credit utilization this month
6	Credit turnover this month
7	Number of bank-initiated payment reversals,i.e. returned cheques/cancelled direct debits in the current month

The bank customers are described by two static features, seven dynamic features and one categorical feature. The first two features are used as unique identification numbers of customers, such as the customer number and account number. The seven dynamic features characterize the state of an account each month and are represented by sequences of 24 measurements. They are summarized in table 15–23:

One of the categorical features provided for bank customers determines special account properties which can be savings / time deposits / depots and can take two values, such as "yes" or "no". According to bank experts, customers with or without these account properties must be treated separately since they may exhibit different payment behavior. Therefore, the data set will be separated into two subsets according to this categorical feature. The first set of customers characterized by feature valus "yes" and possessing a savings account or depots consists of 4,688 customers, while the other set includes 19,579 customers without the said properties of their accounts. These two sets of customers will be denoted hereinafter as groups "Y" and "N", respectively, and the analysis of the customer structure will be performed separately for each group.

After a preliminary analysis of data sets including the calculation of the mean, minimum and maximum values of trajectories and their variances, it can be seen that the value ranges of the seven features are very large and different. The main statistical characteristics of the data group "Y" are summarized in table 15–24.

Those of customers "N" are shown in table 15–25.

The goals of the dynamic analysis of bank customers can be formulated as follows:

1. to find segments of customers with similar payment behavior based on the whole temporal history covering two years;
2. to find segments of customers with similar payment behavior based on the temporal history of half a year, and to follow changes in the cluster structure and in the assignment of customers to the clusters over time.

Table 15–24. Main statistics of each feature of the data group "Y".

Features	*Mean value*	*Standard deviation* σ	$\mu - 3\sigma$	$\mu + 3\sigma$
1	14,510.06	48,097.47	−129,782.34	158,802.46
2	16,347.19	134,581.82	−387,398.26	420,092.63
3	7.77	100,780.11	−302,332.56	302,348.10
4	37,655.46	273,818.07	−783,798.76	859,109.67
5	7,946.71	36,071.66	−100,268.26	116,161.68
6	81,002.74	636,081.44	−1,827,241.60	1,989,247.10
7	0.01	0.00	−0.11	0.12

Table 15–25. Main statistics of each feature of data group "N".

Features	*Mean value* μ	*Standard deviation* σ	$\mu - 3\sigma$	$\mu + 3\sigma$
1	20,811.16	63,966.97	−171,089.75	212,712.06
2	−4,278.04	145,423.24	−440,547.76	431,991.68
3	−13,393.86	153,139.98	−472,813.81	446,026.08
4	7,360.17	157,022.60	−463,707.63	478,427.98
5	17,426.61	123,226.32	−352,252.36	387,105.59
6	35,406.50	318,825.34	−921,069.54	991,882.53
7	0.01	0.00	−0.10	0.11

The first goal can be achieved by clustering customers represented by trajectories of their features on the time interval of two years. The clustering results provide information about the structure within the customer portfolio appearing during this time interval until the current moment. These results are suitable for distinguishing between "good" and "bad" customers according to their long-term payment behavior. The analysis of a long history is often carried out by banks to achieve reliable results, particularly for recognizing "bad" credit customers. The drawback of this analysis is, however, that the classifier cannot be used to classify new observations of existing customers or observations of new customers for the next two years, since the cluster prototypes are described by trajectories with a length of 24 months and thus cannot be compared with shorter sequences of observations. Thus, the classification of new observations and updating the classifier (if necessary) can be repeated every two years. In this case the design of the classifier is static, but the classifier is dynamic in nature since it is applied to dynamic objects.

A more applicable classifier can be designed by clustering sequences of observations over half a year, which is the second goal of the analysis conducted. This analysis allows one to recognize customer segments based on the short-term payment behavior of customers and to detect temporal changes in the customer behavior. The classification of new observations of existing or new customers can be repeated every six months providing up-to-date information about the customers' states and their development. If changes in the customer structure are detected, the classifier is adapted according to the detected changes, which corresponds to an update of the customer segments and their descriptions. Therefore, this type of analysis is based on dynamic classifier design and classification applied to dynamic objects.

The following tasks will be performed, using pointwise similarity as defined in section 13.3.2.

It would exceed the scope of this book to explain in detail the algorithmic steps performed in this study. It should be mentioned, however, that pointwise similarity is defined by

Table 15–26. Scope of the analysis of bank customers.

I. Length of temporal history	*II. Type of customers*	*III. Type of similarity measure for trajectories*
The whole temporal history $t = [1, 24]$	Customers of group 'Y'	Pointwise similarity
Time windows equal to half a year	Customers of group 'N'	

$$\mu(y, a) = \frac{1}{1 + ay^2}$$

which difines the membership function of the fuzzy set "approximately zero", which was described in figure 13–22.

A is defined as a function of α of the α-cut chosen of this fuzzy set and β is a parameter that determines the shape of the membership function

$$\alpha = \frac{1 - \alpha}{\alpha\beta^2}.$$

The arithmetic mean is used for aggregation. For other parameter settings see [Angstenberger 2001]. When segmenting (clustering) customers, it is important that as many as possible of the customers are absorbed (belong to) a cluster and that the customers outside of clusters (here called "stray-customers") are not too numerous. This depends, amongst other parameters, on the α-cut and the number of clusters used. For the customers group "Y" table 15–27 shows the number of stray-customers for different α-cuts (here called u°) and for two clusters. For u° = 0.5 the numbers for 3 and 4 cluster are 692 and 560 respectively.

Table 15–28 shows the respective results for N-customers.

For this group the numbers of stray-customers are 4,563 for c = 3 and 14,270 for c = 4.

Hence, for both groups the number of clusters c = 2 is optimal.

The features used for clustering are now trajectories and not point. The following two figures show this, exemplarily, for the cluster centers of feature 1 for the customer groups "Y" and "N".

So far the segmentation was performed on the basis of the whole 24 month history. A similar analysis was done on the basis of a moving time window of six months with similar results. It showed quite well how customers moved from one cluster to another over time.

Table 15–27. Absorbed and stray customers for "Y"-group.

	Absorbed		
	C_1	C_2	*Stray*
u° = 0.3	3,114	1,419	155
u° = 0.4	3,099	1,396	193
u° = 0.5	3,081	1,367	240
u° = 0.6	3,065	1,308	315

Table 15–28. Absorbed and stray-customers for "N"-group.

	Absorbed		
	C_1	C_2	*Stray*
u° = 0.3	12,227	5,975	1,377
u° = 0.4	11,073	5,964	2,542
u° = 0.5	9,494	5,942	4,143
u° = 0.6	6,479	5,801	7,299

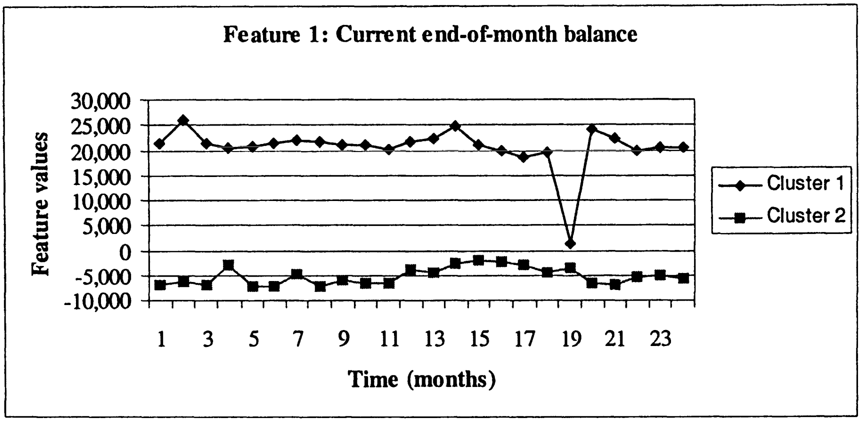


Figure 15–27. Feature 1: current end-of-month balance for "Y".

Tables 15–29 and 15–30 indicate this movement of customers.

After conducting four types of analysis for different customer groups and for different lengths of the temporal history it is necessary to compare the results obtained. It has already been stated that the customer segments recognized based on the whole temporal history and in the first time window are very similar, however the feature values characterizing cluster centers in the first case are somewhat larger in the absolute values compared to those in the second case.

Comparing the results for customers in group "Y" and "N," it can be seen that the values of the end-of-month balance of customers in group "Y" exceed the corresponding values of customers in group "N", and vary in the larger value range. The credit turnover (not shown here) of the first customer group is approximately 10,000–20,000 DM larger than the values of the other customer group,

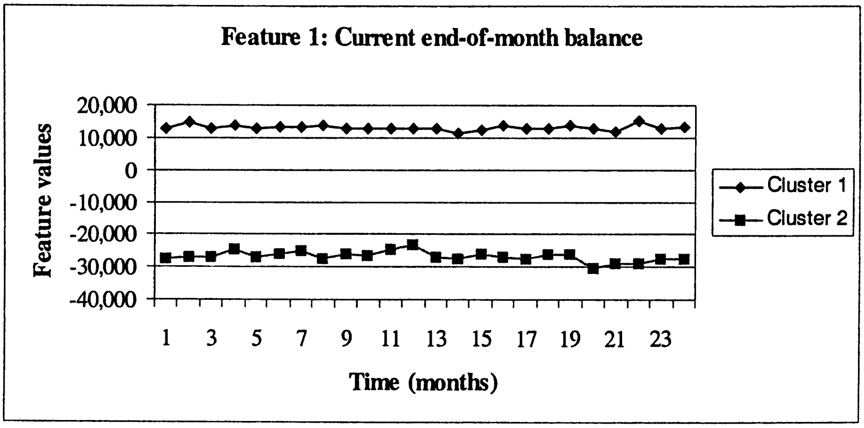


Figure 15–28. Feature 1: current end-of-month balance for "N".

Table 15–29. Temporal change of assignment of customers in group "Y" to clusters.

	From tw_1 to tw_2		*From tw_2 to tw_3*		*From tw_3 to tw_4*	
Number of customers	C_1	C_2	C_1	C_2	C_1	C_2
Remained in C_i	1,118	2,947	1,046	2,921	1,120	2,997
Moved from C_1 into C_2	64		91		45	
Moved from C_2 into C_1	31		102		40	
Dropped out of C_i	55	74	77	82	75	106
Appeared in C_i	94	92	131	65	50	65

Table 15–30. Temporal change of assignment of customers in group "N" to clusters.

	From tw_1 to tw_2		*From tw_2 to tw_3*		*From tw_3 to tw_4*	
Number of customers	C_1	C_2	C_1	C_2	C_1	C_2
Remained in C_i	10,466	4,323	9,951	4,045	10,250	4,321
Moved from C_1 into C_2	108		336		104	
Moved from C_2 into C_1	163		235		123	
Dropped out of C_i	653	501	911	795	612	571
Appeared in C_i	569	644	780	634	672	633

whereas the credit utilization (not shown here) of the active users is 20,000–30,000 DM lower. Therefore, customers in group "Y" belonging to the segment of "active users" have more entries in their accounts, higher monthly account statements and use bank credit less actively than customers in group "N". Customers in the second segment, "non-users", are similar in their behavior for both groups of customers.

The results of analysis conducted in this section can help a bank to better understand the customer portfolio, to distinguish between different groups of active users and non-users in order to be able to develop particular marketing strategy which may be, for instance, offering special favorable services to a group of the most active users.

Bank customer segmentation was carried out based on the dynamic data representing customers' temporal behavior and by applying the dynamic fuzzy clustering algorithm. The dynamic analysis allows to take into consideration the payment behavior of customers over a period of time which characterizes customers much better than a single observation. Until now in most applications related to customer segmentation and described in the literature the static analysis of customers was performed based on measurements at a certain moment of time. These analysis results are obviously not very reliable, since clusters, or customer segments, obtained from such analysis can often change due to significant fluctuations of account feature values that requires periodic reclustering. By contrast the dynamic fuzzy clustering helps to save time and can provide more reliable complete results.

Exercises

1. In what ways and for what purposes can fuzzy sets be used in operations research?
2. Explain why in model 14–3 every nondecreasing operator can be used to combine the goal with all of the constraints.
3. Could approaches (13.9) or (13.18) have been used in model 14–2? If so, what would have been the consequences?
4. In section 14.3.1, a fuzzy decision model has been employed as an optimization criterion. Can this approach be used for both precise and heuristic algorithms?
5. In the system presented in section 14.3.2, the decision support module picks one precedence constraint out of the subset C of the set of all unordered pairs of operations. Consider multiple criteria for the selection of the subset C. Discuss possible fuzzy aggregation models for the derivation of the subset C.

6. Assume in model 14–7 that the instructors' availability is given by the following table:

Instructor number	Weeks 1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	.5	0.5	1	1	1	.5	0	0
8	0	0	.5	1	1	1	1	.5	.5	0	0
9	.5	1	1	1	1	1	1	.5	.5	0	0
11	0	0	0	.5	.5	1	1	1	.5	.5	0
12	1	1	.5	.5	1	1	1	1	1	1	1
13	0	1	1	.5	.5	1	1	1	.5	0	0
17	1	1	1	1	1	1	1	1	1	1	1
21	.5	.5	1	1	.5	0	0	0	0	0	0
23	1	1	1	1	.5	.5	.5	0	0	0	0

Determine a new table 14–15 of available weeks for courses, and try to determine heuristically a first week's final schedule.

7. Discuss approaches, and their advantages and disadvantages, for PERT networks in which activity times are fuzzy and stochastically uncertain.
8. Determine the critical path for the network shown in table 14–14 by substituting for the addition of activity time in the normal critical path method the extended addition (section 5.3.1).
9. Determine an optimal policy for model 14–8 modified as follows: the demands are $a_i = \{40, 40, 45, 50\}$, and the fuzzy set goal is characterized by the membership function

$$\mu_{G_{N+1}} = \begin{cases} 1 - \dfrac{x_{N+1}}{10} & \text{if} \quad 0 \leq x_{N+1} \leq 10 \\ 0 & \text{else} \end{cases}$$

10. In the example shown in table 14–18, the membership degrees of the districtings were evaluated subjectively by the decision maker. Consider fuzzy accessibility measures for the "nearness" or "accessibility" of a service point to every other point in a district for a location problem greater than that shown in figure 14–15. Develop a model in which these accessibility measures are aggregated to a fuzzy measure for the "acceptability" of every district and are further aggregated to a fuzzy measure for the membership

degree of every districting to the fuzzy set of "best districtings." Discuss the sensitivity of such an approach to the choice of the intersection operator.

11. Discuss the possible use of expert systems and FLC model in operations research. Do those approaches satisfy sound OR principles?

16 EMPIRICAL RESEARCH IN FUZZY SET THEORY

16.1 Formal Theories vs. Factual Theories vs. Decision Technologies

The terms *model*, *theory*, and *law* have been used with a variety of meanings, for a number of purposes, and in many different areas of our lives. It is therefore necessary to define more accurately what we mean by models, theories, and laws in order to describe their interrelationships and to indicate their use before we can specify the requirements they have to satisfy and the purposes for which they can be used. To facilitate our task, we shall distinguish between definitions given and used in the scientific area and definitions and interpretations as they can be found in more application oriented areas, which we will call "technologies" in contrast to "scientific disciplines." By technologies we mean areas such as operations research, decision analysis, and information processing, even though these areas call themselves sometimes theories (i.e., decision theory) and sometimes science (i.e., computer science, management science, etc.). This statement is by no means a value judgment; we only want to indicate that the main goals of these areas are different. While the main purpose of a scientific discipline is to generate knowledge and to come closer to truth without making any value judgments, technologies normally try to generate tools for solving problems better, very often

H.-J. Zimmermann, *Fuzzy Set Theory - and Its Applications*

by either accepting or being based on given value schemes. Let us first turn to the area of scientific inquiry and consider the following quotation concerning the definition of the term *model*: "A possible realization in which all valid sentences of a theory T are satisfied is called a model of T."

Harré [1967, p. 86] states, "A model, a, of a thing, A, is in one of many possible ways a replica or an analogue of A." And a few years later, "In certain formal sciences such as logic and mathematics a model for or of a theory is a set of sentences, which can be matched with the sentences in which the theory is expressed, according to some matching rule. . . . The other meaning of 'model' is that of some real or imagined thing or process, which behave similarly to some other thing or process, or in some other way than in its behavior is similar to it" [Harré, 1972, p. 173]. He sees two major purposes of models in science: (1) logical: to enable certain inferences to be made that would not otherwise be possible; and (2) epistemological: to express and enable us to extend our knowledge of the world. Models, according to Harré, are used either as a heuristic to simplify a phenomenon or to make it more readily manageable and explanatory where a model is a model of the real causal mechanism.

Leo Apostel [1961, p. 4] provides us with a very good example for various definitions of models as tuples of a number of components in the following definition: "Let then R (S, P, M, T) indicate the main variables of the modelling relationship. The subject S takes, in view of the purpose P, the entity M as a model of the prototype T." For the four components of the definition, he gives a number of examples that are quite informative concerning the use of models in science and that can be summarized as follows:

Subjects (S) and purposes (P):

1. For a certain domain of facts, let no theory be known. If we replace our study of this domain by the study of another set of facts for which a theory is well known and that has certain important characteristics in common with the field under investigation, then we use a model to develop our knowledge from a zero (or near zero) starting point.
2. For a domain D of facts, we do have a full-fledged theory, but one too difficult mathematically to yield solutions, given our present techniques. We then interpret the fundamental notions of the theory in a model, in such a way that simplifying assumptions can express this assignment.
3. If two theories are without contact with each other, we can try to use the one as model for the other or introduce a common model interpreting both and thus relating both languages to each other.
4. If a theory is well confirmed but incomplete, we can assign a model in the hope of achieving completeness through the study of this model.
5. Conversely, if new information is obtained about a domain, to assure our-

selves that the new and more general theory still concerns our earlier domain, we construct the earlier domain as a model of the later theory and show that all models of this theory are related to the initial domain, constructed as model, in a specific way.

6. Even if we have a theory about a set of facts, this does not mean that we have explained those facts. Models can yield such explanations.
7. Let a theory be needed about an object that is too big or too small, too far away, or too dangerous to be observed or experimented upon. Systems are then constructed that can be used as practical models, experiments that can be taken as sufficiently representative of the first system to yield the desired information.

   Often we need to have a theory present to our mind as a whole for practical or theoretical purposes. A model realizes this globalization through either visualization or realization of a closed formal structure.

Thus, models can be used for theory formation, simplification, reduction, extension, adequateness, explanation, concretization, globalization, action, or experimentation.

Entity (M) and model type (T):

M and T are both images or both perceptions or both drawings or both formalisms (calculi) or both languages or both physical systems. M can also be a calculus and T a theory or language, or vice versa.

> Apostel believes that all models that can be constructed by varying the contents of the four components form a systematic whole: Models are used for system restructuration because of their relations with the system (partial discrepancy); because of their relationship among each other (partial inconsistency at least multiplicity); because of their relationship with themselves (locally inconsistent or locally vague).

By now two things should have become obvious:

1. There is a very large variety of types of models, which can be classified according to a number of criteria. For our deliberation, one classification seems to be particularly important: The interpretation of a model as a "formal model" and the interpretation as a "factual, descriptive model." This corresponds to Rudolph Carnap's distinction between a logical and a descriptive interpretation of a calculus [Carnap 1946]. For him, a logically true interpretation of a model exists if, whenever a sentence is true, the second is equally true and if a whenever a sentence is refutable in the calculus, it is also false in the model. An interpretation is factual interpretation if it is not a logical interpretation, which means that whether a model is true or false does not depend

only on its logical consistency but also on the (empirical) relationship of the sentences (axioms of the model) to the properties of the factual system of which the model is supposed to be an image. The second interpretation of a model is the one that is quite common in the empirical sciences and it is the one we will primarily be referring to in the following.

2. There is certainly a relationship between a model and a theory. This relationship, however, is seen differently by different scientists and by different scientific disciplines. We will now try to specify this relationship because theories, to our mind, are the focal point of all scientific activities.

For Harré [1972, p. 174] "A theory is often nothing but the description and exploitation of some model," or "Development of a theory on the other involves the superimposing of one model on another" [1967, p. 99].

White [1975] eventually simply points out that

> There is a need to logically separate a model and a theory and that they play supporting roles in decision analysis, *viz.*, some theory is needed so that aspects of models can be tested and that some model is needed so that the affects of some changes can be examined. In particular validation of a model needs a theory.

Thus, there seems to be a very intimate relationship between a model and a theory in scientific inquiry. Both, probably to varying degrees, are based on hypotheses, and these hypotheses can either be formal axioms or scientific laws. These scientific laws seem to us to fundamentally distinguish models and theories in scientific disciplines from the type of models (sometimes also called theories) in the more applied areas: "An experimental law, unlike a theoretical statement invariably possesses a determinate empirical content which in principal can always be controlled by observational evidence obtained by those procedures" [Nagel 1969, p. 83].

These laws as scientific laws assert invariance with respect to time and space. The tests to which such hypotheses have to be put before they can claim to be a law depend on the philosophical direction of the scientist. Karl Popper, as probably the most prominent representative of "critical rationalism," believes that laws are only testable by the fact of their falsifiability. Popper holds further that a hypothesis is "corroborated" (rather than confirmed) to the degree of severity of such tests. Such a corroborated hypothesis may be said to have stood up to the test thus far without being eliminated. But the test does not confirm its truth. A good hypothesis in science, therefore, is one that lends itself to the severest test, that is, one that generates the widest range of falsifiable consequences [Popper 1959].

16.1.1 Models in Operations Research and Management Science

The area of operations research will be considered as an example of a more application-oriented discipline, which is here called "technology," in which modeling plays a predominant role. Even though one might dispute whether operations research is a science or a technology, this discussion will follow Symonds, who, as the President of the Institute of Management Science, stated, "Operations Research is the development of general scientific knowledge" [Symonds 1965, p. 385].

What, now, is a model in operations research? Most authors using the term *model* take it for granted that the reader knows what a model is and what it means. Arrow, for instance, uses the term *model* as a specific part of a theory when he says, "Thus the model of rational choice as built up from pairwise comparisons does not seem to suit well the case of rational behaviour in the described game situation" [Arrow 1951]. He presumably refers to the model of rational choice, because the theory he has in mind does not give a very adequate description of the phenomena with which it is concerned, but only provides a highly simplified schema. In the social and behavioral sciences as well as in the technologies, it is very common that a certain theory is stated in rather broad and general terms while models, which are sometimes required to perform experiments in order to test the theory, have to be more specific than the theories themselves. "In the language of logicians it would be more appropriate to say that rather than constructing a model they are interested in constructing a quantitative theory to match the intuitive ideas of the original theory" [Suppes 1961]. Rivett, in his book *Principles of Model Building* [1972], offers three different kinds of classifications of models; when enumerating the models that he suggests be put into the different classes, he no longer uses the term *model* but talks of "problems in this area" and "the theory of this area" as a not-too-well-defined entity of knowledge. Ackoff suggests as a model of decision making a six-phases process that is supposed to be a good picture (model) of the real decision-making process [Ackoff 1962]. This is only one example of quite a number of very similar models of decision making.

If we consider the size of some of the models used in operations research, containing more than 10,000 variables and thousands of constraints, we can easily see what does not distinguish a theory from a model: It is not the complexity, it is not the size, it is not the language, and it is not even the purpose. In fact, there seems to be only a gradual distinction between theory and model. While a theory normally denotes an entire area or type of problem, it is more comprehensive but less specific than a model (e.g., decision theory, inventory theory, queueing theory, etc.); a model most often refers to a specific context or situation and is meant to be a mapping of a problem, a system, or a process. In contrast to a scientific theory, containing scientific laws as hypotheses, a model normally does not assert invariance with respect to time and space but requires modifications whenever the specific context for which the model was constructed changes.

In the following, we will concentrate on models rather than on theories. Realizing that there is quite a variety of types of models, we do not think that it is important and necessary for our purposes to distinguish models by their language (mathematics or logic is considered to be a modeling language), by area, by problem type, by size, and so on. One classification, however, seems to be important: the distinction of models by their character. Scientific theories were already divided into formal theories and factual theories. For models, particularly in the area of the technology in which values and preferences enter our considerations, we will have to distinguish among the following:

1. *Formal models.* These are models that are purely axiomatic systems from which we can derive if–then statements and the hypotheses of which are purely fictitious. These models can only be checked for consistency; they can neither be verified nor falsified by empirical arguments.
2. *Factual models.* These models include in their basic hypotheses falsifiable assumptions about the object system; that is, conclusions drawn from these models have a bearing on reality and they, or their basic hypotheses, have to be verified or can be falsified by empirical evidence.
3. *Prescriptive models.* These are models that postulate rules according to which processes have to be performed or people have to behave. This type of model will not be found in science, but it is a common type of model in practice.

The distinction between these three different kinds of models is particularly important when using them: All three kinds of models can look exactly the same, but the "value" of their outputs is quite different. It is therefore rather dangerous not to realize which type of model is being used, because we might take a formal model to be a factual model or a prescriptive model to be a factual model, and this could have quite severe consequences for the resulting decision.

As an example, let us look at the above-mentioned Ackoff model of decision making. Is it a formal, a factual, or a prescriptive model? If it is a formal model, we cannot derive from it any conclusion for real decision making. If it is a factual model, then it would have to be verified or falsified before we can take it as a description of real decision making. The assertion, however, that decision making proceeds in phases was already empirically falsified in 1966 [Witte 1968]. Still, a number of authors stick to this type of model. Do they want to interpret their model as a prescriptive model? This would only be justified if they could show that, for instance, decision making can be performed more efficiently when done in phases. This, however, has never been shown empirically. Therefore, we can only conclude that authors suggesting a multiphase scheme as a model for decision making take their suggestion as a formal model and do not want to make any statement about reality, or that they are using a falsified, that is, invalid and false, factual model.

16.1.2 Testing Factual Models

The quality of a model depends on the properties of the model and the functions for which the model is designed. In general, models will have to have at least the following three major properties: logical consistency, usefulness, and efficiency. By *logical consistency*, we mean that all operations and transformations have been performed properly and that all conclusions follow from the hypothesis. This consistency has to be demanded of all types of models, whether they are formal, factual, or prescriptive. By *usefulness*, we mean that the model has to be helpful for the function for which it has been designed. By *efficiency*, we mean that the model, as the tool to achieve an end, has to fulfill the desired function at a minimum of effort, time, and cost.

In decision making and problem solving, factual models will be needed to describe, to explain, and to predict phenomena and consequences. For "conditional predictions," formal models will also be useful in order to obtain if–then statements, for instance, in the framework of simulation. Formal models will also be useful and necessary for the area of communication within the decision-making process and for relaying the resolutions or conclusions of the decision or problem-solving process to the "actors." One should assume that prescriptive models are the most common in decision making. This, however, is only true if one calls all "decision models"—that is, models that contain an objective function by which an optimal solution can be determined—prescriptive models. To our mind, this is not quite appropriate because these kinds of models only prepare suggestions for possible decisions; the normative or prescriptive character is acquired only after the "solution" has been declared a decision by the authorized decision maker. A much more important feature of these models, it seems to us, is that they have to describe or define properly the conditions that limit the action space (such as capacities, financial resources, legal restrictions, etc.).

We can now restate the notion of the quality of a model more precisely: we already mentioned that consistency is one of the necessary conditions for quality. Usefulness of a model will have to be defined for each of the three different types of models differently:

1. While a factual model can be called useful, if it is "factually true" (by contrast to logically true), that is, if it maps the object system with an appropriate precision (which can only be tested empirically), the model also has to generate knowledge—that is, the user of a model should gain knowledge he or she would not have gained without using the model or which he or she did not have available before using the model.
2. Formal models can be neither verified nor falsified empirically. Such a model will be considered useful if activities such as teaching, explaining, and communication become more efficient with the model than without it.
3. Prescriptive models also cannot be verified or falsified. They are the more useful the more effectively they help to enforce the desired behavior, to control predefined performance measured, and to define ranges within which decision makers have freedom to decide.

Two prime factors in modeling are the modeling language and the quality of input data. The type of modeling language appropriate for models in decision making was already discussed in chapter 1. Here we shall elaborate some more on the quality of input data.

The saying "garbage in—garbage out" is well known and speaks for itself. The following quotation from Josiah Stamp [1975, p. 236] points in the same direction: "Governments are very keen in amassing statistics. They collect them, add them, raise them to the *n*th power, take the cube root and make wonderful diagrams. But what you must never forget is that every one of these figures comes in the first instance from the village watchman who just puts down what he damn pleases."

It must, however, be borne in mind that the effort put into deriving and obtaining numerical values or relations must be geared to the value of the model, and that when data are scarce it may still be useful to draw conclusions from not fully satisfactory input data. In this case, a tentative look at the dependence of the solution upon the quality of the input data may be very advisable.

The quality of the input data is closely related to the question of operational definitions for the relevant variables. The processes of defining variables and their operational indicators and measurement are intertwined. To quote White [1975, p. 102], "We take 'measurement' to be a special aspect of a 'definition'." One might take the view that measurement is the actual procedure for assigning the real numbers that constitute the measure. However, as pointed out in a previous section, this is the quantification process and in itself does not constitute a measure unless it is a homomorphism. The homomorphism then defines the measure. Very often when modeling in the area of social sciences, one will find that relations, data, or values are stated in very vague ways. Goals, for instance, may be stated as "trying to achieve satisfactory profits," data as "the South of the

Table 16–1. Hierarchy of scale levels.

	Permissible transformation			
Type of scale	*Verbal*	*Formal*	*Invariance*	*Example*
Nominal scale	One-to-one function	$x_i \neq x_j \to x'_i \neq x'_j$	Uniqueness of values	License plates
Ordinal scale	Monotonic increasing function	$x_i \leq x_j \to x'_i \leq x'_j$	Rank order of values	Marks
Interval scale	Affine function	$x' = a \cdot x + b$	Ratio of differences	Temperature (C°, F°)
Ratio scale	Similarity function	$x' = a \cdot x$	Ratio of values	Length (cm, inch)
Absolute	Identity	$x' = x$	Values	Frequency

country is much poorer than the North," and relations as "his investment strategies were much more risky than those of his competitors." Very often these variables are measured subjectively, and point scales are used to transform the "measurements" into numerical values. Even though it is necessary to include in the model variables that are considered important but that are very hard to operationalize and measure, the quality of the input data might have very limiting effects on the degree of transformation of these variables that can be permitted in the model. Rather than neglecting these kinds of data, one should consciously determine which scale quality these data have and then make sure that only admissible transformations are being used when processing these data in the model. Table 16–1 sketches the hierarchy of scale levels including the permissible transformations for each of the levels.

The testability of the components of a model—in the scientific and in the practical context—depends largely on the operational definition of the hypotheses. In this sense, observation and formal analysis prior to model building can very often improve the testability of hypotheses. Let us illustrate this with the following example. In decision analysis, one normally distinguishes among decision making under certainty, decision making under risk, and decision making under uncertainty. One assumes that in decision making under risk the decision maker is able to store and process probability distribution functions. Here probabilities ought to be interpreted as Koopman-type probabilities—that is, probabilities as expressions of belief rather than in the frequentistic sense. This hypothesis is hardly testable because a situation of decision making under risk is not homogenous with respect to the available information at all. An improvement in the testability of hypotheses could be achieved if one would distinguish, for instance, among the following:

1. Decision making when quantitative probabilities are known (intervalscale)
2. Decisions when interval probabilities are known (hyperordinal scale)
3. Decisions when qualitative probabilities are known (ordinal scale)
4. Decisions when partially ordered nominal probabilities are known (ordinal scale)
5. Decisions when nominal probabilities are known (states are known but not truth ratable)
6. Decisions when only some of the nominal probabilities are known

It is obvious that the information storage and processing requirements that a human would need in order to decide "rationally" are quite different in the above cases and that the permissible operations in the model will also be different depending on the type of probability that can be assumed to exist.

If the testing is done on the basis of the outputs of the analysis, the decision maker might already be able to indicate that the output of the analysis is not satisfactory, probably because important relations or variables have been omitted. If the decision maker or expert rates the output of the model as satisfactory, it gains the status of face-validity, sometimes in practice the most we can hope for.

Ideally a model should now be tested by implementation, that is, by comparing actual with predicted results. This, however, in many instances is impossible for several reasons.

1. *Changes of environment:* Factors such as sales, price levels, and so on might have changed while the model was built and implemented, and therefore the observed results after implementation of the model can no longer be compared with the predicted results.
2. *Changes in performance:* If, for instance, the model is tested after implementation by running the old procedure parallel to the model and if the old procedure included human activities, the performance of these activities might be improved by the persons because they know that the "new" model is being compared with their performance, which would probably drop again, if and when the operation of the new procedures would be terminated.
3. *Risk and uncertainty:* It is obvious that if procedures have been designed to optimally decide in situations of risk or uncertainty, the "real" results cannot meaningfully be compared with the probabilistic prediction.
4. *Optimality:* If only one solution is actually implemented, there is, of course, no way to compare this with other alternatives. In many cases, the optimal solution with which the model solution could be compared is not known at

all because it is not computable or because optimality was defined subjectively in a way that is not objectively reproducible.

It has already been pointed out that all kinds of theories and models can be and ought to be tested for consistency. In formal analysis, it might even be possible to prove consistency, which does not mean that models and theories for which consistency has not yet been proven are not formally correct. For "factual" or "substantial" theories and models, empirical testing of basic hypotheses, relations, and resulting outputs is absolutely necessary in order to achieve a certain degree of confirmation of the theory or the model. This fact is often neglected when working with theories and models. If, for instance, the hypothesis of "rationality" in decision-making models is "justified" by defining rationality by more basic axioms such as transitivity, reflexibility, existence of an ordering, and so on, which seem quite plausible and natural, then the model or the theory might become more testable but certainly not better confirmed. To confirm the model would require empirically testing either the main hypothesis or the presumably more operational basic axioms. This, of course, still does not determine uniquely the methods that can be used for testing hypotheses. These methods will depend on the area in which the model is being used (physics, engineering, management) and the purposes for which the model has been built. Thus, in scientific inquiry, probabilistic tests might not be acceptable because scientific laws assert deterministic invariance. These methods, however, might be the only available ones for testing models in areas such as management, sociology, and political decision making.

In the following we shall report on empirical research concerning two main components of fuzzy set theory: Membership functions and operators (connectives, aggregators).

16.2 Empirical Research on Membership Functions

Measurement means assigning numbers to objects such that certain relations between numbers reflect analogous relations between objects. In other words, measurement is the mapping of object relations into numerical relations of the same type.

If it is possible to prove that there is a homomorphic mapping $f\colon E \to N$ from an empirical relational structure $\langle E, P_1, \ldots, P_n\rangle$ with a set of objects E and an n-tuple of relations P_i into a numerical relational structure $\langle N, Q_1, \ldots, Q_n\rangle$ with a set of numbers N and relations Q_i, then a scale $\langle\langle E, N, f\rangle\rangle$ exists. By specifying the admissible transformations, the grade of uniqueness is determined.

Therefore measurement starts by formulating the properties of the empirical structure; implicitly, the intended object space is modeled on a nonnumerical level. Strictly speaking, at the very beginning there should be a semantic definition of the central concepts; this would considerably facilitate the consistent use of the relevant principles. Unfortunately, this definition has not yet been possible for the concept of membership. Membership has a clear-cut formal definition. However, explicit requirements for its empirical/experimental measurement are still missing. Under these circumstances, it is not surprising that apart from first steps by Norwich and Turksen [1981], genuine measurement structures have not yet been developed.

Under these circumstances, one could wait and see, until a satisfactory definition is available. However, one should remember that up to the beginning of the twentieth century, even in the "hard sciences," measures were used without being equipped with adequate measurement theories. Usually the measurement tools used were based on not much more than plausible reasons. Nevertheless, the success of the natural sciences is undisputed. Hence, for the purpose of empirical research, it may be tolerable to use plausible techniques.

Firstly, such a scale can serve as an operational definition of membership. Secondly, a specific concept can be criticized and hence may help to obtain useful improvements. We shall present two models for membership functions. Let us call the first "Type A-model" and the second "Type B-model."

16.2.1 Type A-Membership Model

Of prime importance is the determination of the lowest necessary scale level of membership for a specific application. The purpose of the model A-membership was to empirically investigate aggregation operators. In this instance, it was sufficient to determine degrees of membership for a predefined set of objects rather than continuous membership functions. The requested scale level should be as low as possible in order to facilitate data acquisition, which usually involves the participation of human beings. On the other hand, a suitable numerical handling is desirable in order to insure mathematically appropriate operating. Regarding the five classical scale types—nominal, ordinal, interval, ratio, and absolute scale—the interval scale level seems to be most adequate. In this respect, we cannot follow Sticha, Weiss, and Donnell [1979], who assert that membership has to be measured on an ordinal scale. Usually the intended mathematical operations require at least interval-scale quality.

The easiest way to obtain data is to ask some subjects directly for membership values. However, it is well known that scales that are developed by using the so-called direct methods may be distorted by a number of response biases [Cronbach 1950]. On the other hand, indirect methods work on the basis of much

Table 16–2. Empirically determined grades of membership.

Stimulus x		$\mu_M(x)$	$\mu_C(x)$	$\mu_{M \cap C}(x)$
1.	bag	0.000	0.985	0.007
2.	baking tin	0.908	0.419	0.517
3.	ballpoint pen	0.215	0.149	0.170
4.	bathtub	0.552	0.804	0.674
5.	book wrapper	0.023	0.454	0.007
6.	car	0.501	0.437	0.493
7.	cash register	0.692	0.400	0.537
8.	container	0.847	1.000	1.000
9.	fridge	0.424	0.623	0.460
10.	Hollywood swing	0.318	0.212	0.142
11.	kerosene lamp	0.481	0.310	0.401
12.	nail	1.000	0.000	0.000
13.	parkometer	0.663	0.335	0.437
14.	pram	0.283	0.448	0.239
15.	press	0.130	0.512	0.101
16.	shovel	0.325	0.239	0.301
17.	silver spoon	0.969	0.256	0.330
18.	sledgehammer	0.480	0.012	0.023
19.	water bottle	0.564	0.961	0.714
20.	wine barrel	0.127	0.980	0.185

weaker assumptions using ordinal judgments only. Their advantages are simplicity and robustness with respect to response biases.

Their disadvantage is that many judgments are needed, since the ordinal judgment provides relatively little information. This drawback seemed acceptable in order to avoid distortions of the data. Thus we decided to use a method that yields an interval scale on the basis of ordinal ratings: After a set of suitable objects has been established, subjects are asked for the grades of membership on a percentage scale. People are accustomed to this type of judgment, and division by 100 provides the normalized 0–1 values. The obtained data are interpreted as ranks. The subsequent scaling procedure refers mainly to a method suggested by Diederich, Messick, and Tucker [1957] based on Thurstone's "Law of Categorical Judgment" [Thurstone 1927].

A detailed description of the method can be found in Thole, Zimmermann, and Zysno [1979]. Table 16–2 illustrates the type of membership information that was obtained and the type of objects used for experimentation. The transformation of the observed information to degrees of membership was performed by a computer program written for this purpose.

16.2.2 Type B-Membership Model

Often a certain concept can be considered as a context-specific version of a more general feature. For instance, the set of young men is a subset of all objects with the feature age. We shall call this general feature the "base variable." This coincides with the definition of a base variable in definition 9–1. The scale of the base variable that is normally generally accepted (here age in years) will be called a "judgmental scale." In contrast to the scale of the base variable, the scale of the "specific version" is context-dependent. Thus a term in definition 9–1 does not necessarily correspond to "the specific version" of the base variable, because "terms" did not explicitly assume a specific context. If the term *young* refers to the age of men (by contrast to the age of flies, cars, houses, or dinosaurs), then we can assume that the observer has some idea about what "young" means with respect to men. He has a "standard" with respect to which he evaluates age in terms of "young," "old," etc. We shall, therefore, call this specific context-dependent scale an "evaluational scale." If there exist a judgmental scale and an evaluational scale, both referring to the same empirical relational structure, then a mapping from one numerical relative into the other that reflects the differences of the basic empirical relational structure with respect of the same set of elements would be possible. If, on the other hand, the scale of (for instance) the base variable and the mapping (function) was known, then the scale of the special feature could be determined. The mapping (function) can be considered as the membership function, which has to be determined. The required scale level of the membership function essentially remains the same as for type A model. In contrast to model A, however, we used direct scaling methods. These involve less effort and are justified by the existence of the base variable, which provides extra control with respect to judgmental errors of the subjects. The judgmental (valuation) of membership can be regarded as the comparison of object x with a standard (ideal), which results in a distance $d(x)$. If the object corresponds fully with the standard, the distance shall be zero; if no similarity between standard and object exists, the distance shall be "∞." If the evaluation concept is represented formally by a fuzzy set $\tilde{P} \subset X$, then a certain degree of membership $\mu_{\tilde{P}}(x)$ is assigned to each element x. We shall assume that this degree of membership is a function of the "distance," d, between the two above-mentioned scales ($\tilde{P}$ representing a fuzzy set defined context-dependently as a subset of the universe X).

Thus we define

$$\mu_{\tilde{P}}(x) = \frac{1}{1+d(x)} \tag{16.1}$$

where $d(x)$ is the "distance" of the two scales for the element $x \in X$. The distance function now has to be specified. A specific monotonic function of the similarity with the ideal could, as a first approximation, be $d'(x) = 1/x$.

Experience shows, however, that ideals are very rarely fully realized. As an aid to determine the relative position, very often a context-dependent standard b is created. It facilitates a fast and rough preevaluation such as "rather positive," "rather negative," and so on. As another context-dependent parameter, we can use the evaluation unit a, similar to a unit of length such as feet, meters, yards, and so on. If one realizes furthermore that the relationship between a physical unit and perceptions is generally exponential [Helson 1964], then the following distance function seems appropriate:

$$d(x) = \frac{1}{e^{a(x-b)}} \tag{16.2}$$

Substituting equation (16.2) into model (16.1) yields the logistic function

$$\mu_{\tilde{P}}(x) = \frac{1}{1+e^{-a(x-b)}} \tag{16.3}$$

It is S-shaped, as demanded by several authors [Goguen 1969; Zadeh 1971]. Formally, b is the inflexion point and a is the slope of the function.

From the point of view of linear programming, model (16.3) has the additional advantage that it can easily be linearized by the following transformation:

$$-\ln\frac{1-\mu}{\mu} = \ln\frac{\mu}{1-\mu} = a(x-b) \tag{16.4}$$

where μ stands for $\mu_{\tilde{P}}(x)$.

The parameters a and b will have to be interpreted differently depending on the situation that is modeled. From a linguistic point of view, a and b can be considered as semantic parameters.

Model (16.3) is still too general to fit subjective models of different persons. Frequently only a certain part of the logistic function is needed to represent a perceived situation. This is also true for measuring devices such as scales, thermometers, and so on, which are designed for specific measuring intervals only.

In order to allow for such a calibration of our model, we assume that only a certain interval of the physical scale is mapped into the open interval (0, 1) (see

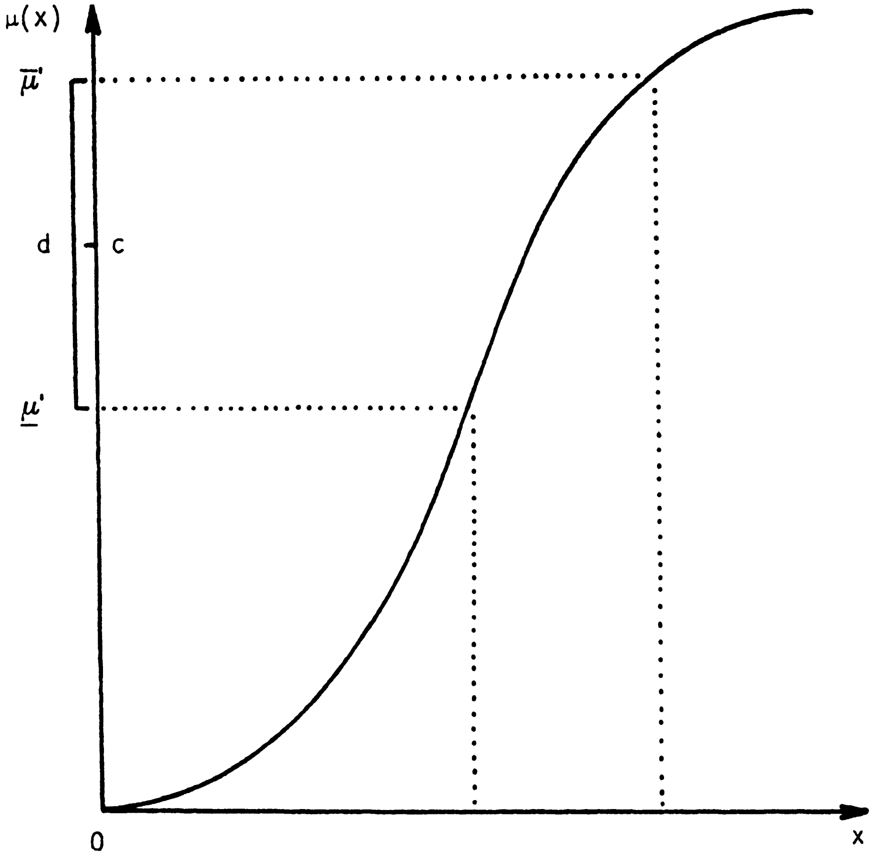


Figure 16–1. Calibration of the interval for measurement.

figure 16–1). Whenever stimuli are smaller than or equal to the lower bound or larger than or equal to the upper bound, the grade of membership of 0 or 1, respectively, is assigned to them. This is achieved by changing the range by legitimate scale transformations such that the desired interval is mapped into [0, 1].

Since we requested an interval scale, the interval of the degrees of membership may be transformed linearly. On this scale level, the ratios of two distances are invariant. Let $\bar{\mu}$ and $\underline{\mu}$, respectively, be the upper and lower bounds of the normalized membership scale, let μ_i be a degree of membership between these bounds, $\underline{\mu} < \mu_i < \bar{\mu}$, and let $\underline{\mu}'$, μ_i', $\bar{\mu}'$ be the corresponding values on the transformed scale. Then

$$\frac{\mu_i - \underline{\mu}}{\overline{\mu} - \underline{\mu}} = \frac{\mu_i' - \underline{\mu}'}{\overline{\mu}' - \underline{\mu}'} \tag{16.5}$$

For the normalized membership function, we have $\underline{\mu} = 0$ and $\overline{\mu} = 1$.
Hence

$$\mu_i' = \mu_i(\overline{\mu}' - \underline{\mu}') + \underline{\mu}' \tag{16.6}$$

Generally it is preferable to define the range of validity by specifying the interval d with the center c as shown in figure 16–1.
Hence

$$\overline{\mu}' = d + \underline{\mu}' \tag{16.7}$$

and

$$\underline{\mu}' = 2c - \overline{\mu}' \tag{16.8}$$

Substituting equation (16.7) into equation (16.8) yields

$$\underline{\mu}' = 2c - d - \underline{\mu}' \tag{16.9}$$

Solving equation (16.9) for $\underline{\mu}'$ gives

$$\underline{\mu}' = c - d/2 \tag{16.10}$$

and inserting equations (16.10) and (16.7) into equation (16.6) yields

$$\mu_i' = d(\mu_i - 1/2) + c \tag{16.11}$$

The general model of membership (16.3) is specified by two parameters of calibration, if μ_i is replaced by μ_i'. Solving this equality for μ_i leads to the complete model of membership:

$$\mu_i = \overset{0}{\Big\lceil} \left(\frac{1}{1 + e^{-a(x-b)}} - c \right) \frac{1}{d} + \frac{1}{2} \overset{1}{\Big\rceil} \tag{16.12}$$

$\overset{0}{\lceil} \cdot \overset{1}{\rceil}$ indicates that values outside of the interval [0, 1] have no real meaning. The measurement instrument does not differentiate there. Hence

$$\begin{aligned} x < \underline{x} &\rightarrow \mu(x) = 0 \\ x > \overline{x} &\rightarrow \mu(x) = 1 \end{aligned} \tag{16.13}$$

The determination of the parameters from empirical databases does not pose any difficulties in the general model (16.3). It should be mentioned that not only monotonic functions, such as those discussed so far, can be described, but so can unimodal functions—by representing them by an increasing (S_I) and a decreas-

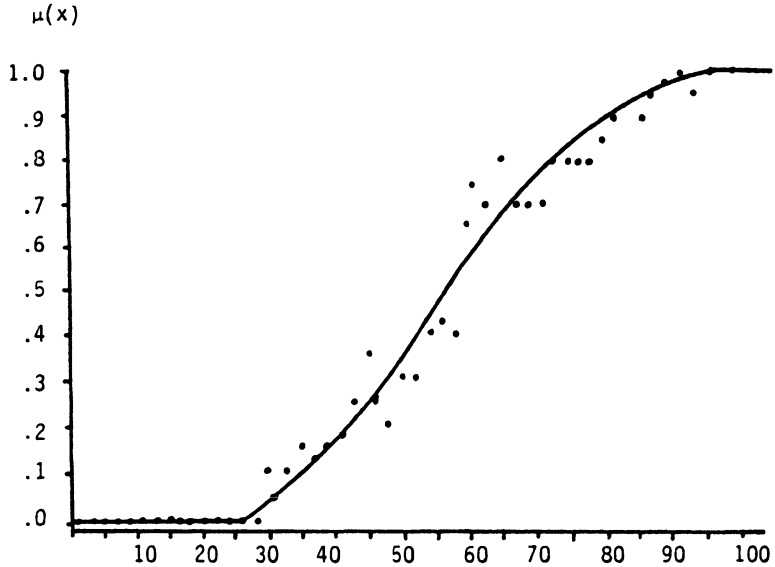


Figure 16–2. Subject 34, "Old Man."

ing (S_D) part. Formally, they can be represented as the minimum or maximum, respectively, of two monotonic membership functions each:

$$\mu_{S_I S_D}(x) = \min\left[\overset{0}{\mu_{S_I}(x)}, \overset{1}{\mu_{S_D}(x)}\right]$$
$$\mu_{S_I S_D}(x) = \max\left[\overset{0}{\mu_{S_I}(x)}, \overset{1}{\mu_{S_D}(x)}\right]$$

A computer program was written to process the observed data.

The type B-model for membership functions, which provides a membership function rather than degrees of membership for single elements of a fuzzy set (as Type A does), was also empirically tested.

We shall present results concerning a very common fuzzy set, "young men," "old men," and so on. Having available membership functions, we could also test models of modifiers such as "very."

The evaluation of the data showed a good fit of the model. Figures 16–2 through 16–7 show the membership functions given by six different persons. As can be seen, the concepts "very young men" and "young men" are realized in the monotonic type as well as in the unimodal. The detailed data and results can be found in a major report of the authors [Zimmermann and Zysno 1982].

One may ask whether a general membership function for each of the four sets can be established. Though the variety of conceptual comprehension is rather

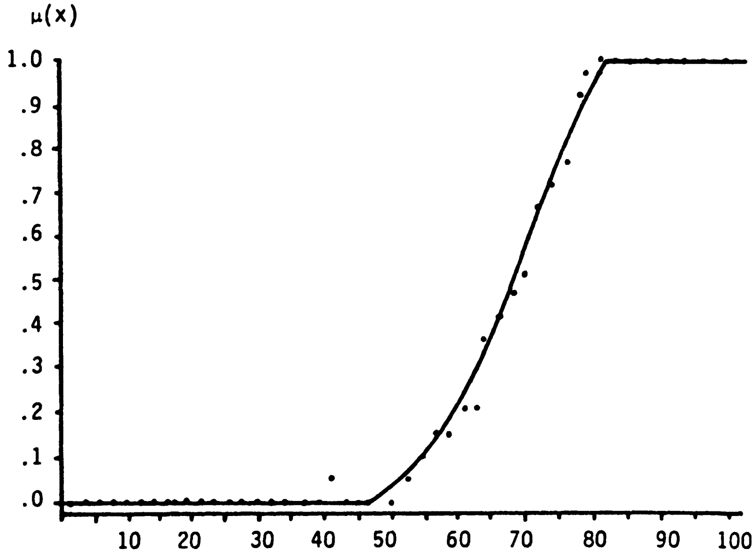


Figure 16–3. Subject 58, "Very Old Man."

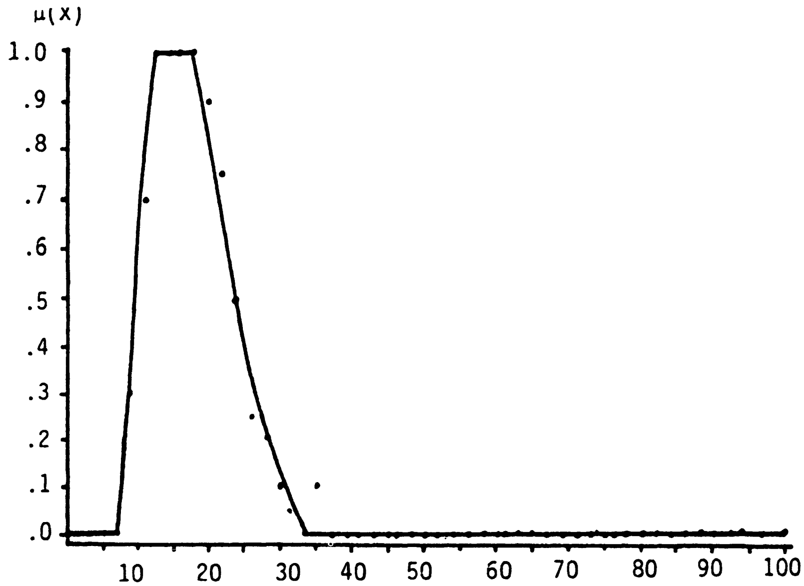


Figure 16–4. Subject 5, "Very Young Man."

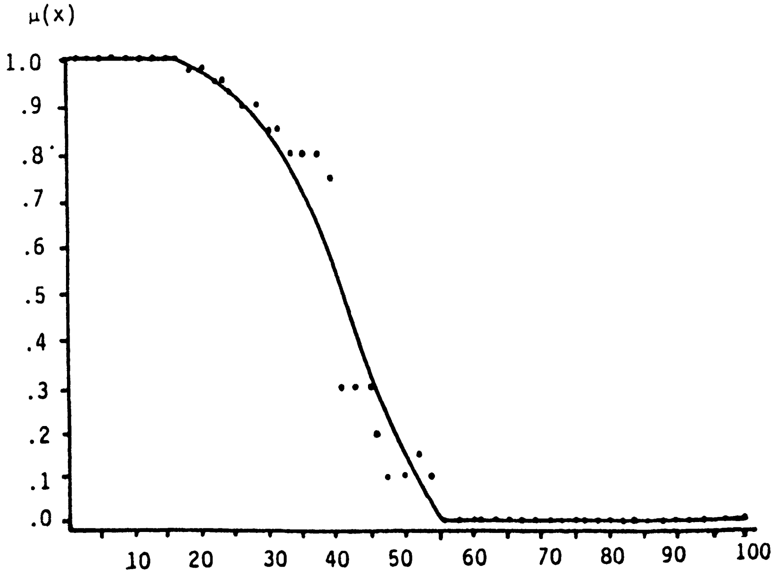


Figure 16–5. Subject 15, "Very Young Man."

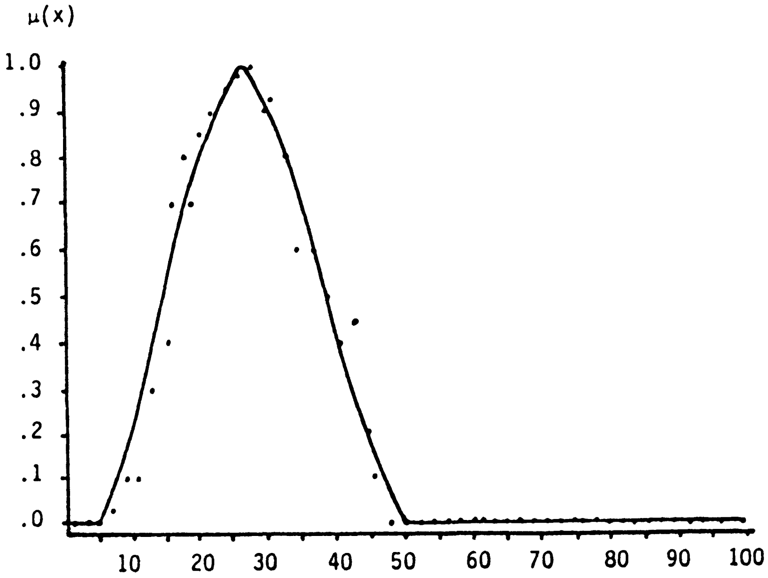


Figure 16–6. Subject 17, "Young Man."

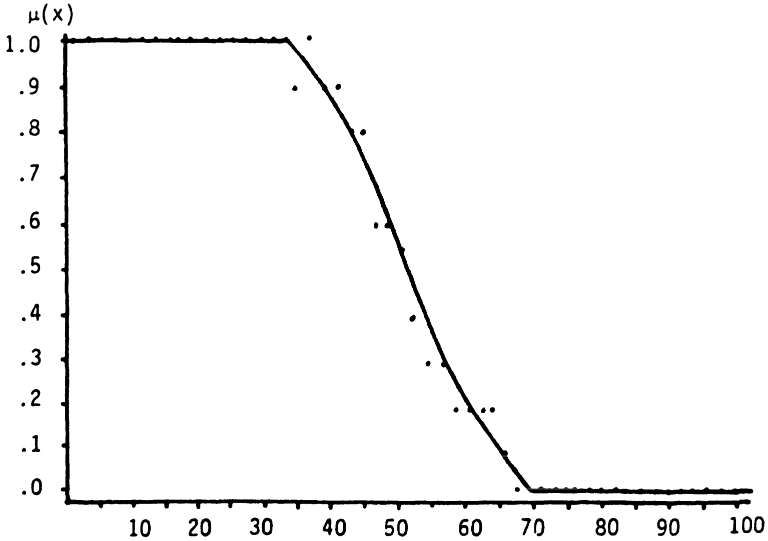


Figure 16–7. Subject 32, "Young Man."

remarkable, there should be an overall membership function at least in order to have a standard of comparison for the individuals. This is achieved by determining the common parameter values *a*, *b*, *c*, and *d* for each set. Obviously, the general membership functions of "old man" and "very old man" are rather similar (see figures 16–8 and 16–9). Practically, they differ only with respect to their inflection points, indicating a difference of about five years between "old man" and "very old man." The same holds for the monotonic type of "very young man" and "young man"; their inflection points differ by nearly 15 years. It is interesting to note that the modifier "very" has a greater effect on "young" than on "old," but in both cases it can be formally represented by a constant. Several subjects provided the unimodal type in connection with "very young" and "young." Again the functions show a striking congruency.

16.3 Empirical Research on Aggregators

In section 3.2.2, a number of possible operators were mentioned. We saw that they were assigned in various ways to set-theoretic operations, such as intersection, union, etc. For some of these operators, axiomatic formal justifications were also given. In definition 14–1, the triple decision-intersection-min-operator

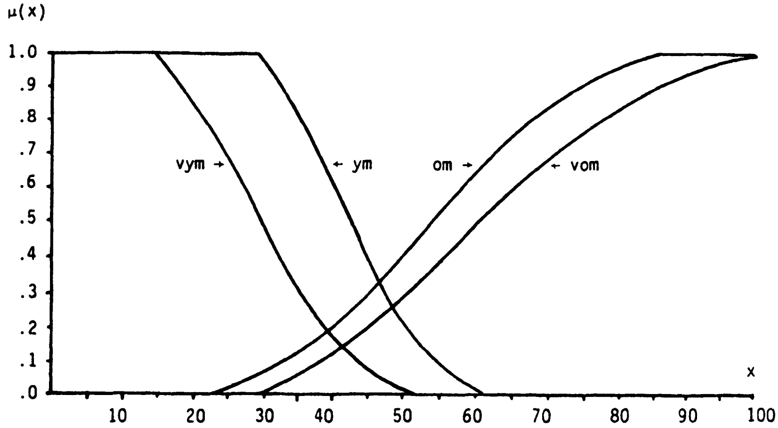


Figure 16–8. Empirical membership functions “Very Young Man,” “Young Man,” “Old Man,” “Very Old Man.”

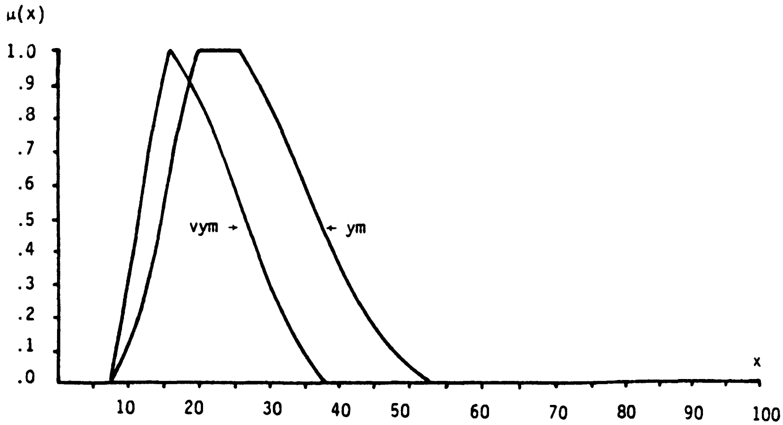


Figure 16–9. Empirical unimodal membership functions “Very Young Man”, “Young Man.”

was used. Some indication was given there that, from a factual point of view, this triple might turn out not to be true. After what has been said in section 16.1, it should be obvious that for a factual use of fuzzy set models only empirical verification of models for the aggregators is appropriate. This can only be done in specific contexts, and the results will therefore be of limited validity.

Some empirical testing of aggregators has been performed in the context of fuzzy control. We shall report on empirical research done in the context of human evaluation and decision making, that is, concerning the question, "How do human beings aggregate subjective categories, and which mathematical models describe this procedure adequately?"

As already mentioned, the term *decision* has been defined in many different ways. A decision also has many different aspects, for example, the logical aspect, the information-processing aspect, etc. We shall focus our attention on the last aspect: The search for and the modeling, processing, and aggregation of information. A decision in the sense of definition 14–1, rather than being some kind of optimization, is the search for an action that satisfies all constraints and all aspiration levels representing goals. "Deciding" about the creditworthiness of a person might be called an "evaluation" rather than a decision. It means, however, checking on whether a person satisfies all aspiration levels concerning security, liquidity, business behavior, and so on.

In the following, we will give a rough description of two experiments and their results. The first experimental design started from the triple "decision-intersection-min-operator" and tried to find out whether the min-operator was adequate for modeling the intersection. However, it did not question the pair "decision-intersection." The second experiment is no longer limited to considering a decision as the intersection; it relinquishes the set-theoretic interpretation of a decision altogether.

Test 1: Intersection-min-operator [Thole et al. 1979]

Two fuzzy sets, $\tilde{A}$ and $\tilde{B}$, were considered. It seems reasonable to demand that the following conditions concerning the judgmental "material" are satisfied:

1. The attributes characterizing the members of the sets $\tilde{A}$ and $\tilde{B}$ are independent, that is, some magnitude of $\mu_{\tilde{A}}$ is not affected by some magnitude of $\mu_{\tilde{B}}$ and vice versa. As an operational criterion for this kind of independence, a correlation of zero is demanded:

$$r_{\mu_{\tilde{A}}\mu_{\tilde{B}}} = 0$$

2. If $\mu_{\tilde{A}\cap\tilde{B}}$ represents the aggregation of $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, modeling the intersection, and if $w_{\tilde{A}}$ and $w_{\tilde{B}}$ are weights, then $\mu_{\tilde{A}\cap\tilde{B}}$ can be described by

$$\mu_{\tilde{A}\cap\tilde{B}} = (w_{\tilde{A}}\mu_{\tilde{A}}) \circ (w_{\tilde{B}}\mu_{\tilde{B}})$$

Where $^\circ$ stands for some algebraic operation. But since the models proposed do not take into account the different importance of the sets with respect to their intersection, equal weights are demanded:

$$w_{\tilde{A}} = w_{\tilde{B}}$$

As an operational criterion for equal weights, equal correlations are demanded:

$$r_{\mu_{\tilde{A}}\mu_{\tilde{A}\cap\tilde{B}}} = r_{\mu_{\tilde{B}}\mu_{\tilde{A}\cap\tilde{B}}}$$

With regard to these conditions, three fuzzy sets were chosen: "metallic object" [*Metallgegenstand*], "container" [*Behälter*], and "metallic container" [*Metall-behälter*].[1] It has to be proved that these sets satisfy the conditions mentioned above.

Now the following hypotheses may be formulated: Let $\mu_{\tilde{M}}(x)$ be the grade of membership of some object x in the set "metallic object" and $\mu\ (x)$ be the grade of membership of x in the set "container"; then the grade of membership of x in the intersection set "metallic container" can be predicted by

$$H_1:\quad \mu_{\tilde{M}\cap\tilde{C}}(x) = \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{C}}(x)\}$$

$$H_2:\quad \mu_{\tilde{M}\cap\tilde{C}}(x) = \mu_{\tilde{M}}(x) \cdot \mu_{\tilde{C}}(x)$$

A pretest was carried out in order to guarantee that these assumptions were justified.

Sixty students at the RWTH Aachen from 21 to 33 years of age, all of them native speakers of the German language, served as unpaid subjects in the main experiment. Each subject was run individually through two experimental sessions, the first one taking about 20 minutes, the second one about 40 minutes. In order to eliminate influences of memory as much as possible, the interviews were performed at an interval of approximately three days.

Each subject was asked to evaluate each of the objects with respect to being a member of $\tilde{A}$ (metallic object), $\tilde{B}$ (container), and $\tilde{A} \cap \tilde{B}$ (metallic container). The three resulting membership scales are shown in table 16–2.

Now, what about the prediction of the empirical data for "metallic container" by the two candidate rules? Table 16–3 shows the empirical results together with the grades of membership computed by using the min-operator and the product-operator, respectively.

Figures 16–10 and 16–11 show graphically the relationship between empirical and theoretical grades of membership. The straight line indicates locations of perfect prediction—that is, if the operator makes perfect predictions and the data are free of error, then all points lie on the straight line.

[1] This investigation has been carried out in Germany. The corresponding German word is given in brackets. It should be realized that the German language allows the forming of compound word; hence the intersection is labeled by one word.

Table 16–3. Empirical vs. predicted grades of membership.

Stimulus x	$\mu_{\tilde{M}\cap\tilde{C}}(x)$	$\mu_{\tilde{M}\cap\tilde{C}}(x) \mid min$	$\mu_{\tilde{M}\cap\tilde{C}}(x) \mid prod.$
1. bag	0.007	0.000	0.000
2. baking tin	0.517	0.419	0.380
3. ballpoint pen	0.170	0.149	0.032
4. bathtub	0.674	0.552	0.444
5. book wrapper	0.007	0.023	0.010
6. car	0.493	0.437	0.219
7. cash register	0.537	0.400	0.252
8. container	1.000	0.847	0.847
9. fridge	0.460	0.424	0.264
10. Hollywood swing	0.142	0.212	0.067
11. kerosene lamp	0.401	0.310	0.149
12. nail	0.000	0.000	0.000
13. parkometer	0.437	0.335	0.222
14. pram	0.239	0.283	0.127
15. press	0.101	0.130	0.067
16. shovel	0.301	0.293	0.078
17. silver spoon	0.330	0.256	0.248
18. sledgehammer	0.023	0.012	0.006
19. water bottle	0.714	0.546	0.525
20. wine barrel	0.185	0.127	0.124

The question arises: Are the observed deviations small enough to be tolerable? To answer this question we chose two criteria:

1. if the mean difference between observed and predicted values is not different from zero ($\alpha = 0.25$; two-tailed), and
2. if the correlation between observed and predicted values is higher than 0.95, the connective operator in question should be accepted.

Since the observed differences are normally distributed, we used the student t = test as a statistic. It is entered by the mean of the population (in this case, 0), the mean of the sample (0.052 for the min-operator and 0.134 for the product-operator), the observed standard deviation (0.067 for the minimum and 0.096 for the product), and the sample size (20). For the min-rules, the result is $t = 3.471$, which is significant ($df = 19$; p, the probability of transition, is less than 0.01). For the product rule, the result is $t = 6.242$, which is also significant ($df = 19$; p is less than 0.001). Thus, both hypotheses H_1 and H_2 have to be rejected.

Despite the fact that none of the connective operators tested seems to be a really suitable model for the intersection of subjective categories, there is a slight superiority of the min-rule, as can be seen from the figures. If one were forced

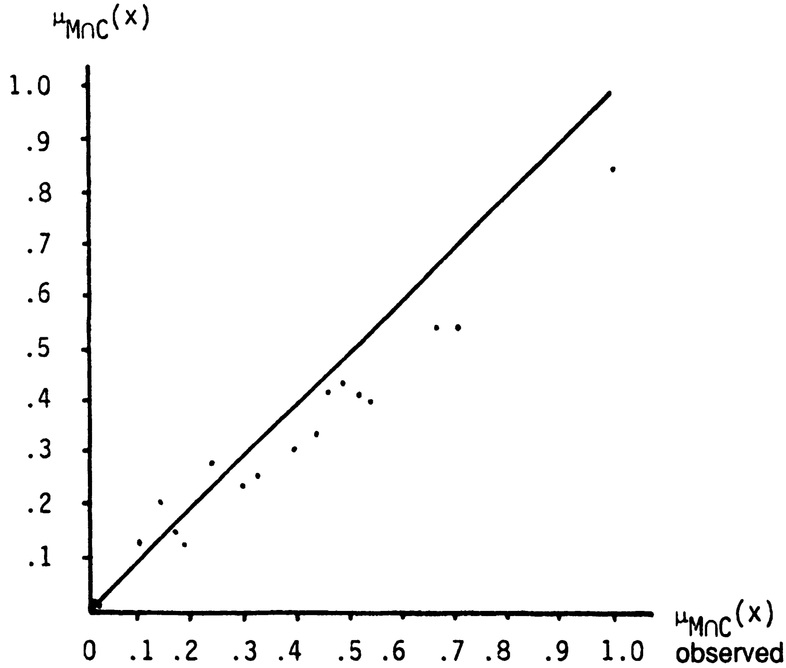


Figure 16–10. Min-operator: Observed vs. expected grades of membership.

to use one of these aggregation rules, then the minimum certainly would be the better choice.

The results of this experiment indicate that both product and minimum fail to be perfect models for the intersection operation in human categorizing processes.

Test 2 [Zimmermann and Zysno 1980]

The interpretation of a decision as the intersection of fuzzy sets implies no positive compensation (trade-off) between the degrees of membership of the fuzzy sets in question if either the minimum or the product is used as an operator. Each of them yields degrees of membership of the resulting fuzzy set (decision) that are on or below the lowest degree of membership of all intersecting fuzzy sets (see test).

The interpretation of a decision as the union of fuzzy sets, using the max-operator, leads to the maximum degree of membership achieved by any of the fuzzy sets representing objectives or constraints. This amounts to a full compen-

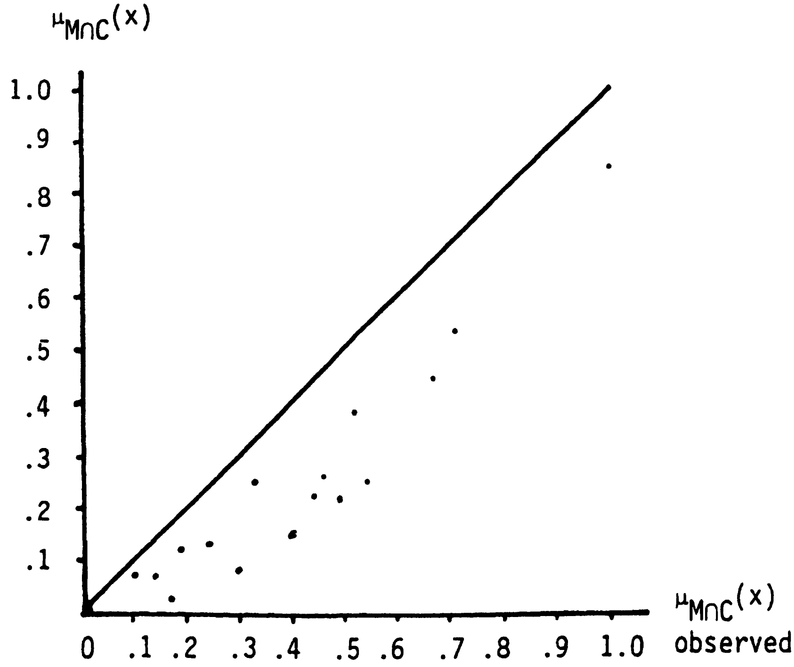


Figure 16–11. Product-operator: Observed vs. expected grades of membership.

sation of lower degrees of membership by the maximum degree of membership (see example 14–4).

Observing managerial decisions, one finds that there are hardly any decisions with no compensation between different degrees of goal achievement or between the degrees to which restrictions are limiting the scope of decisions. The compensation, however, rarely seems to be "complete," as would be assumed using the max-operator. It may be argued that compensatory tendencies in human aggregation are responsible for the failure of some classical operators (min, product, max) in empirical investigations.

Two conclusions can probably be drawn: Neither the noncompensatory "and" represented by operators that map between zero and the minimum degree of membership (min-operator, product-operator, Hamacher's conjunction operator (see definition 3–15), Yager's conjunction operator (see definition 3–16) nor the fuzzy compensatory "or" represented by operators that map between the maximum degree of membership and 1 (maximum, algebraic sum, Hamacher's disjunction operator, Yager's disjunction operator) are appropriate to model the aggregation of fuzzy sets representing managerial decisions. It is necessary to define new additional operators that imply some degree of compensation, that is, that map also between the minimum degree of membership and the maximum degree of membership of the aggregated sets. In contrast to modeling the non-compensatory "and" or the fully compensatory "or," they should represent types of aggregation that we shall call "compensatory and."

It is possible that human beings use many nonverbal connectives in their thinking and reasoning. One type of these connectives may be called "merging connectives," which may be represented by the "compensatory and." Being forced to verbalize them, people may possibly map the set of "merging connectives" into the set of the corresponding language connectives ("and," "or"). Hence, when talking, they use the verbal connective they feel to be closest to their "real" nonverbal connective.

In analogy to the verbal connectives, the logicians defined the connectives $\wedge$ and $\vee$, assigning certain properties to each of them. By this, compound sentences can be examined for their truth values. In contrast to this constructive process, the empirical researcher has to analyze a given structure. Therefore, in order to induce subjects to use their own connectives, we avoided the verbal connectives "and" and "or" in our experiment, but tried to ask for combined membership values implicitly presenting a suitable experimental design and instruction, respectively.

We shall not describe in detail the experimental work in which different compensatory operators were tested and in which the γ-operator (see definition 3–19) turned out to perform best. The reader is referred to Zimmermann and Zysno [1980] for details. We shall return to figure 1–1 and explain how credit clerks arrive at a decision concerning the creditworthiness of customers by aggregating their judgments concerning the determinants of creditworthiness. For details, see Zimmermann and Zysno [1983]. A number of possible compensatory and noncompensatory models were tested.

Searching for an appropriate decision situation, our choice fell on the rating of creditworthiness for the following reasons:

1. This is a decision problem that is complex enough though it is still relatively transparent and definable. In addition, this situation is highly standardized. Even though test subjects come from different organizations, similar evaluation schemes can be assessed.
2. A sufficiently large number of decision makers is available with about the same training background and similar levels of competence.
3. The decision problem to be solved can be formulated and presented in a realistic manner with respect to contents and appearance.

First, the creditworthiness hierarchy shown in figure 1–1 was developed together with 18 credit clerks.

Testing the predictive quality of the proposed models required a suitable basis of stimuli that were to be rated with respect to the creditworthiness criteria and a weighting system that allowed a differentiated aggregation of these criteria.

The natural basis of information for evaluating creditworthiness is the credit file. Therefore, we would have liked to analyze original bank files. However, a selection of finished cases is always a biased sample, since the initially rejected applicants are missing. Moreover, we wanted to avoid unnecessary troubles with banking secrecy. Therefore, it was decided to prepare 50 fictitious applicants for credit.

A credit application form usually contains about 30 continuous or discrete attributes of applicants. If each variable were dichotomized, 2^{30} different borrowers could be produced. Clearly, one cannot realize all possible variations. Therefore, a sample was drawn that satisfied the following two conditions: The 50 applicants (stimuli) should

1. be distributed as evenly as possible along the continuum of each aspect, and
2. be typical for consumer credits.

The files were produced in three stages:

1. One hundred and twenty applications were completed randomly with respect to the grade of extension of the 30 attributes.
2. The resulting 30×120 data matrix was purged of 40 cases most unlikely and least typical. The remaining 80 files were completed using information of an inquiry agency (Schufa) and a short record of a conversation between the client concerned and a credit clerk.
3. The applicants should represent the variability of the eight concepts. If each aspect is dichotomized into two classes ($\mu \leqslant 0.5 \rightarrow 0$, $\mu > 0.5 \rightarrow 1$), then the resulting $2^8 = 256$ patterns of evaluation can be put in a 16×16 matrix. With the assistance of two credit experts, the 80 credit files were placed into this tableau. Finally, 30 files were eliminated in order to obtain equal frequencies in rows and columns.

We could now expect that the 50 applicants varied evenly along each attribute and each criterion. Only one attribute was constant: the credit amount was fixed at DM 8,000, because the judgment "creditworthy" is only meaningful with respect to a certain amount. A borrower might be good for DM 8,000, but not for DM 15,000.

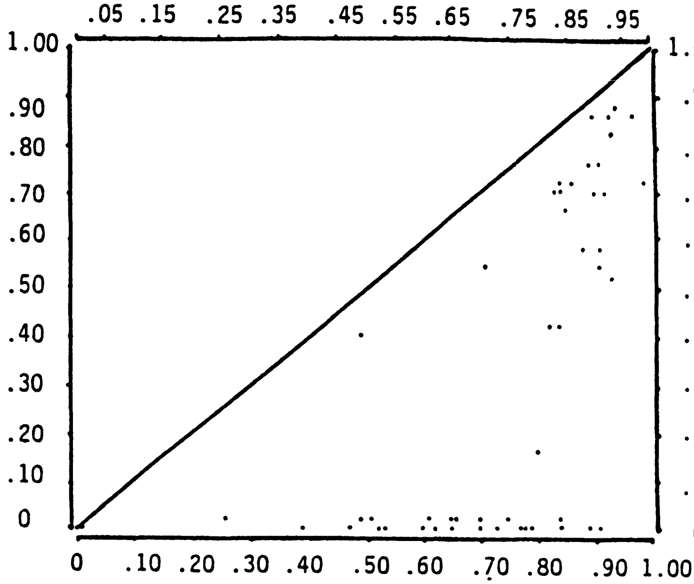


Figure 16–12. Predicted vs. observed data: Min-operator.

Surely it would be interesting to include the credit amount as a variable in this investigation. But in order to receive a stable basis for scaling and interpretation, a serious enlargement of the sample of credit experts would be necessary. This, however, would have considerably exceeded our budget.

The predictive quality of each model can be evaluated by comparing observed μ-grades with theoretical μ-grades. The latter can be computed for higher-level concepts by aggregation of the lower-level concepts using the candidate formula. The membership values for higher-level concepts should be predicted sufficiently well by any lower level of the corresponding branch. The quality of a model can be illustrated by a two-dimensional system, the axes of which represent the observed versus theoretical μ-values. Each applicant is represented by a point. In the case of exact prognosis, all points must be located on a straight diagonal line. As our data are corrected empirically, there will be deviations from this ideal. Figures 16–12 to 16–15 depict some of the typical results of the tests for security as being determined by fourth-level determinants.

Unfortunately, the weighted geometric mean fails drastically in predicting security by unmortgaged real estate and other net properties. In our view, this is due to the fact that the model does not regard different grades of compensation. The inclusion of different weights for the concepts does not seem to be sufficient

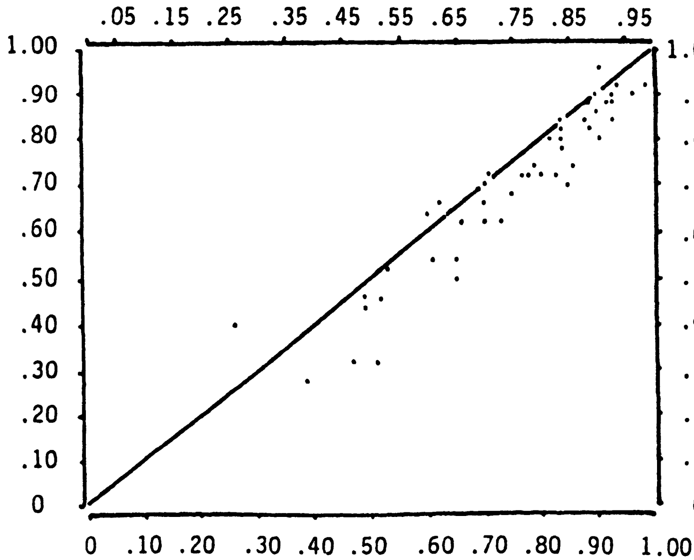


Figure 16–13. Predicted vs. observed data: Max-operator.

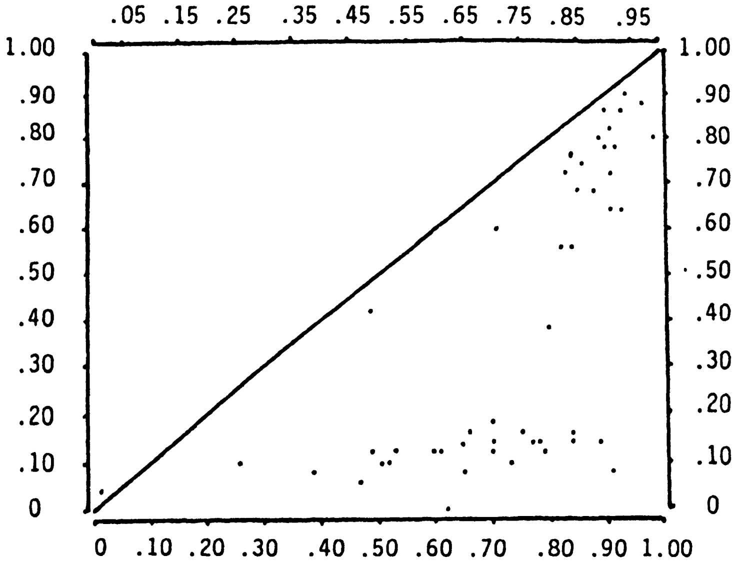


Figure 16–14. Predicted vs. observed data: Geometric mean operator.

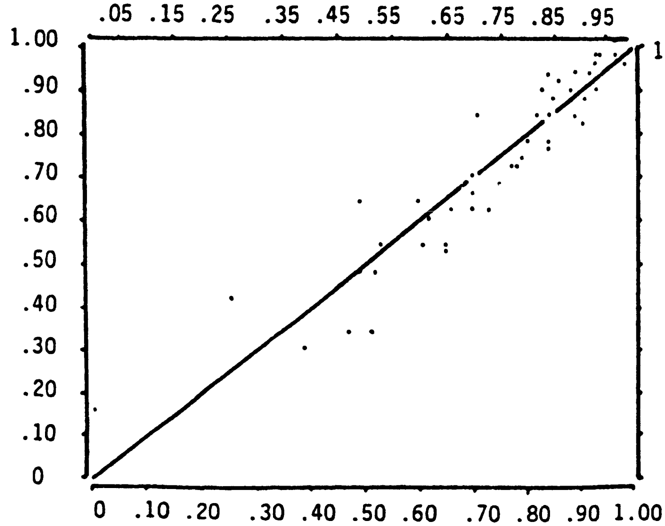


Figure 16–15. Predicted vs. observed data: γ-operator.

for describing the human aggregation process adequately. Consequently, it comes as no surprise that the γ-model, comprising different weights as well as different grades of compensation, yields the best results.

It should be kept in mind, however, that γ has not been determined empirically. This would have required a further experimental study, based on a theory describing the dependence of γ-values between higher and lower levels. For the present, we are content with estimations derived from the data. At least it has been shown that the judgmental behavior of credit clerks can be described quite well if this parameter is taken into account.

Finally, the complete hierarchy of creditworthiness is presented together with the elaborated weighting system and the γ values for each level of aggregation (figure 16–16).

16.4 Conclusions

Our example analysis of the process of rating creditworthiness yields a criteria structure that is concept oriented and self-explanatory. The γ-model, which was from the beginning designed to satisfy mathematical requirements as well as to describe human aggregation behavior, proved most adequate with respect to prog-

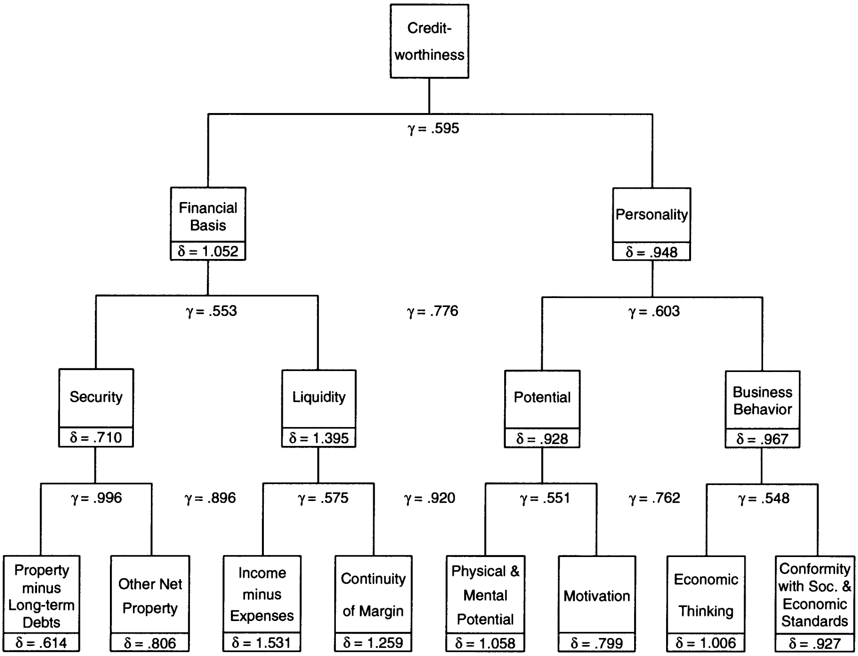


Figure 16–16. Concept hierarchy of creditworthiness together with individual weights δ and γ-values for each level of aggregation.

nostic power. This class of operators is continuous, monotonic, injective, commutative, and in accordance with classical truth tables, which manifests their relationship to formal logic and set theory. They aggregate partial judgments such that the formal result of the aggregation ought to make them attractive for empirically working scientists and useful for the practitioner.

Banking managers not only evaluate but also decide. In order to complete the description of a decision process, we therefore had asked the managers to arrive at a decision for each fictitious credit application. If the creditworthiness were an attribute of the all-or-none type and all credit managers followed the same decision-making process, then two homogenous blocks of credit decisions (one block with 100% yes decisions and one block with 100% no decisions) would result. The number of positive decisions, however, varied over the entire range from 45 to 0. Obviously, there existed a considerable individual decision space.

17 FUTURE PERSPECTIVES

In the first nine chapters of this book, we covered the basic foundations of the theory of fuzzy sets as they can be considered today in an undisputed fashion. Many more concepts and theories could not be discussed, either because of space limitations, because they cannot yet be considered ready for a textbook, or they are too specific and advanced for the goal of this textbook. It was already mentioned in the preface, that now-a-days more than 30,000 publications in the area of fuzzy set theory and computational intelligence exist. It is obvious that they cannot all be covered in such a textbook. I hope, however, that after studying this book the reader will be in a position to read, understand and evaluate most of the papers and books that are being published now. Hopefully the reader has also obtained some feeling how and to what type of problems this technology can be applied.

Fuzzy set theory is certainly not a philosopher's stone that solves all the problems that confront us today. But it has considerable potential for practical as well as for mathematical applications, the latter of which have not been discussed at all in this book.

To indicate the scope of future applications of fuzzy set theory, we shall point to some of the most relevant subject areas. Researchers have become more and more conscious that we should be less certain about uncertainty than we have

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been in the past. The management of uncertainty—that is, uncertainty due to lack of knowledge or evidence, due to an abundance of complexity and information, or due to the fast and unpredictable development of scientific, political, social, and other structures nowadays—will be of growing importance in the future.

In fact, in practice the "fuzzy epoch" has already begun. There already exist quite a number of expert systems and expert-system shells that use fuzzy sets either in the form of linguistic variables or in the inference process (see [Gupta and Yamakawa 1988a]). Fuzzy computes were exhibited as early as 1987 in Tokyo. Gupta and Yamakawa [1988a] provide a very good description of the present state of development.

One of the advantages of fuzzy set theory is its extreme generality, which will enable it to accommodate quite a number of the new developments necessary for coping with existing and emerging problems and challenges. Some areas are already well developed, such as possibility theory [Dubois and Prade 1988a], fuzzy clustering, fuzzy control, fuzzy mathematical programming, etc. Other areas, however, have still ample space for further development.

The area in which primarily fuzzy set theory is known and attractive to many scientists, students and practitioners was certainly fuzzy control. Excellent books, as, for instance, [Babuska 1988] and [Verbruggen et al. 1999] indicate extremely well the present state of this area. Unluckily the attractiveness of this area has to a large extent hidden the other potentials of fuzzy set theory. We hope that the reader of this book has become aware of all the other and not yet exploited possibilities to use this theory in many areas.

Considerably more research—formal as well as empirical—will be necessary in order to cope with these challenges. Much of this research will only be possible through interdisciplinary team efforts. Let us indicate some of the research that is needed. Fuzzy set theory can be considered as a modeling language for vague and complex formal and factual structures. So far, mainly the min-max version of fuzzy set theory has been used and applied, even though many other connectives, concepts, and operations have been suggested in the literature. Membership functions generally are supposed "to be given". Therefore, much empirical research and good modeling effort is needed on connectives and on the measurement of membership functions to be able to use fuzzy set theory adequately as a modeling language. Great opportunities, not yet exploited, exist in the field of artificial intelligence. Most of the approaches and methods offered there so far have been dichotomous. If artificial intelligence really wants to be useful in capturing human thinking and perception, the phenomenon of uncertainty will have to be modeled much more adequately than has been done so far. Here, of course, fuzzy set theory offers many different opportunities.

Very recently an even younger promising application area has emerged: that of web-technology. Large masses of data and information are being made available without improving the human capability of perceiving complex structures in detail. Intelligent agents, data mining, etc. might help to bridge this gap and fuzzy technology will undoubtedly find an almost yet untouched field of research and application. Also new areas, such as ecology, nuclear engineering, etc., have already shown to have large potentials for fuzzy sets.

Another (at least potential) strength of fuzzy set theory is its algorithmic, computational promise. The more we realize that there are problems—the reader might, for instance, think of NP-complete problem structures, which are far too complex for existing traditional approaches (combinatorial programming, etc.) to cope with—the more the need for new computational avenues becomes apparent.

In recent years fuzzy systems have been used to solve, for instance, efficiently systems of differential equations (see [Bardossy 1996]) and one also finds some other applications of that type in recent issues of fuzzy sets and systems.

In general, however, fuzzy set theory has not yet proved to be computationally able to solve large and complex problems efficiently. Reasons for this are that for computation, either we still have to resort to traditional techniques (linear programming, branch and bound, traditional inference) or the additional information contained in fuzzy set models makes computations excessively voluminous. Here prudent standardization (support fuzzy logic, etc.) as well as good algorithmic combinations of heuristics and fuzzy set theory might offer some real promise. In other words, research in the direction of fuzzy algorithms is also urgently needed.

Decision analysis has since 1970 been one of the prominent application areas of fuzzy set theory. In this comprehensive textbook only one chapter could be dedicated to this area. More details can be found in my book "Fuzzy Sets, Decision Making and Expert Systems" [1987, third printing 1993] and other books and papers listed in the bibliography. It is hoped that further research efforts will advance this area and help to close still existing gaps.

Abbreviations of Frequently Cited Journals

ECECSR	Economic Computation and Economic Cybernetics Studies and Research
EIK	Elektronische Informationsverarbeitung und Kybernetik
FJOR	European Journal of Operational Research
FSS	Fuzzy Sets and Systems
JMAA	Journal of Mathematics, Analysis and Applications
J.Op.Res.Soc.	Journal of the Operational Research Society

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